Lecture 21: Coordinates in Hilbert space

updated 18/10

Given an integral pair $(X, \int_X)$ and a choice of scalars $\mathbb{F}$, we have now constructed the Hilbert space $L^2(X,\mathbb{F})$ with pairing

$$\left\langle \lim_{n\to\infty} f_n, \lim_{n\to\infty} g_n \right\rangle = \lim_{n\to\infty} \int f_n \overline{g_n}.$$

From the isomorphism of normed spaces

$$\begin{aligned} L^2(X,\mathbb{F}) &\xrightarrow{\cong} \overline{L^2(X,\mathbb{F})^\vee} \qquad (*) \\ g &\longmapsto \langle -, g \rangle \end{aligned}$$

we have seen how (at least in the case $X = [a,b]$ and $X = S^1$, but it works more generally) integrable functions on X give functionals on $L^2(X,\mathbb{F})$ and hence by the self-duality $(*)$ of Hilbert space, to vectors in L^2. Technically it is convenient to view vectors in L^2 as Cauchy sequences of <u>continuous</u> functions, but we now also have the option to view these vectors as <u>integrable</u> functions modulo almost everywhere equality. The L^2-space achieves the sought for "unification" of integration with function spaces. Recall that this unification was motivated in Lecture 17 by the realisation that the standard tools of linear algebra are not sufficiently powerful to allow us to work with the infinite-dimensional spaces $\mathrm{Cts}(X,\mathbb{R})$.

For example, we know from Stone–Weierstrass (Corollary L16-5, Ex. L17-1) that the functions $\sin(n\theta), \cos(n\theta)$ form a linearly independent set which spans a dense subspace in $\mathrm{Cts}(S^1,\mathbb{R})$, that is

$$\mathrm{Cts}(S^1,\mathbb{R}) = \overline{\mathrm{span}(\{1\} \cup \{\cos(n\theta), \sin(n\theta)\}_{n>0})}.$$

In principle this suggests we understand everything there is to know about continuous functions on the circle. Then, we sit down with a single example $f: S^1 \longrightarrow \mathbb{R}$ and find we cannot say anything about its coefficients in this "dense basis" of trigonometric functions. So actually we know (close to) nothing, in practice! (not an unfamiliar state of affairs for a mathematician, sadly). Let us now fix this sorry mess.

We switch to complex coefficients, because it is right to do so. Set $S^1 := [0, 2\pi]/\sim$.

Lemma L21-1 The set $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is a linearly independent set in $Cts(S^1, \mathbb{C})$ which spans a dense subspace (with respect to the compact-open topology).

Proof To see $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is linearly independent we differentiate a linear dependence relation $\sum_{n=-N}^{N} \mu_n e^{in\theta} = 0$ and evaluate at $\theta = 0$ to find

$$\sum_{n=-N}^{N} (in)^k \mu_n = 0 \qquad k \geq 0$$

This shows the vector of μ_n's is in the kernel of a $(2N+1)\times(2N+1)$ Vandermonde matrix (see the solution of Ex. L17-1) hence zero, so $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is LI.

There is a homeomorphism $Cts(S^1, \mathbb{C}) \overset{\phi}{\cong} Cts(S^1, \mathbb{R}) \times Cts(S^1, \mathbb{R})$ sending a complex-valued function f to $(Re(f), Im(f))$, where on both sides we use the compact open topology (Ex. L12-14). To see that $V = span_{\mathbb{C}}(\{e^{in\theta}\}_{n\in\mathbb{Z}})$ is dense, observe $\cos(n\theta) = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \in V$ and similarly $\sin(n\theta) \in V$. By Corollary L16-5 the set $A = \{1\} \cup \{\cos(n\theta), \sin(n\theta)\}_{n>0}$ is dense in $Cts(S^1, \mathbb{R})$ and hence $A \times A$ is dense in $Cts(S^1, \mathbb{R})^2$ (Ex. L18-9). Since $\phi(V) \supseteq A \times A$ and ϕ is a homeomorphism, this shows V is dense. $\square$

Giving S^1 the default integral pair, the canonical map $\mathrm{Cts}(S^1,\mathbb{C}) \longrightarrow L^2(S^1,\mathbb{C})$ is $\mathbb{C}$-linear and injective so it is immediate from the above that $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is linearly independent in $L^2(S^1,\mathbb{C})$. It is <u>almost</u> immediate that the subspace spanned is dense, but we have to contend with the difference between $\|-\|_2$ and $\|-\|_\infty$ (the latter being associated to the compact-open topology).

<u>Lemma L21-2</u> $L^2(S^1,\mathbb{C}) = \overline{\mathrm{span}_{\mathbb{C}}(\{e^{in\theta}\}_{n\in\mathbb{Z}})}$

<u>Proof</u> We know $\mathrm{Cts}(S^1,\mathbb{C})$ is $\|-\|_2$-dense in $L^2(S^1,\mathbb{C})$ (by construction) and that $V = \mathrm{span}_{\mathbb{C}}(\{e^{in\theta}\}_{n\in\mathbb{Z}})$ is $\|-\|_\infty$-dense in $\mathrm{Cts}(S^1,\mathbb{C})$ by Lemma L21-1, and it suffices to prove V is $\|-\|_2$-dense (because then a closed subset of $L^2(S^1,\mathbb{C})$ containing V must contain $\mathrm{Cts}(S^1,\mathbb{C})$, since $Y \cap \mathrm{Cts}(S^1,\mathbb{C})$ is $\|-\|_2$-closed and contains V, and therefore must be all of $L^2(S^1,\mathbb{C})$). But given $f\in \mathrm{Cts}(S^1,\mathbb{C})$ and $\varepsilon>0$, if $p\in V$ and $\|p-f\|_\infty < \frac{\varepsilon}{\sqrt{2\pi}}$ then by Ex. L18-15 with $p=2, q=\infty$

$$\|p-f\|_2 \le \sqrt{2\pi}\,\|p-f\|_\infty < \varepsilon$$

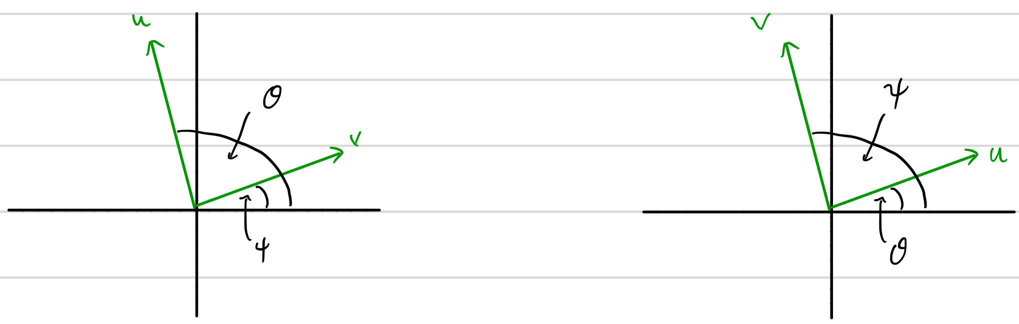
which proves the claim. $\square$

Next we compute the pairings $\langle e^{im\theta}, e^{in\theta}\rangle$ in $L^2(S^1,\mathbb{C})$. But first let us examine what it means more generally to compute $\langle f,g\rangle$ for $f,g\in L^2(X,\mathbb{C})$, and before that, let us recall why we know $\langle,\rangle$ in $\mathbb{R}^2$ is connected to angles.

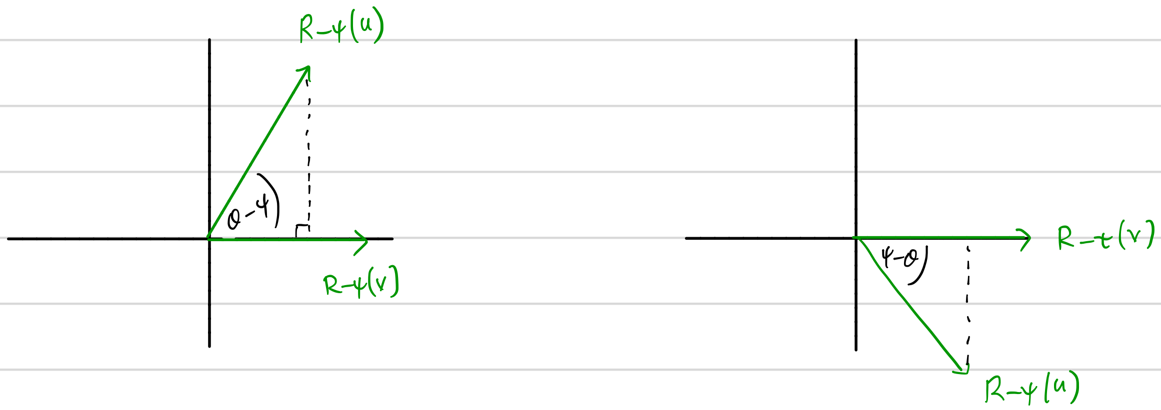
So let $\langle,\rangle$ denote the standard inner product on $\mathbb{R}^2$, so that $O(2)$ is the set of linear transformations T of the plane satisfying $\langle Tu, Tv\rangle = \langle u,v\rangle$ for all $u,v\in\mathbb{R}^2$ and $SO(2) = \{R_\theta\}_{\theta\in\mathbb{R}} \subseteq O(2)$ is the subgroup of rotations.

Suppose u,v are nonzero and they make angles θ,ψ with the x-axis (measuring counter-clockwise, with $0\le\theta,\psi<2\pi$).

Before (case $\theta \geq \psi$) (case $\theta \leq \psi$)



After rotation by $-\psi$



We compute that

$$\begin{aligned} \langle u,v \rangle &= \langle R_{-\psi}(u), R_{-\psi}(v) \rangle \\ &= \langle R_{-\psi}(u), \|v\| \cdot e_1 \rangle \\ &= \|v\| \cdot (R_{-\psi}(u))_1 \\ &= \|v\| \cdot \|u\| \cos(\theta - \psi) \end{aligned}$$

Note that we get the cosine of the angle between u, v, since the full $O(2)$-group preserves the pairing, and two observers on "opposite sides of the plane" (i.e. with different orientations) disagree about whether the oriented (i.e. counter-clockwise) angle between u, v is $\theta - \psi$ or $\psi - \theta$ (see Ex. L3-5).

We saw in Lecture 1 (see p. 7) that two observers at the same position and with the <u>same orientation</u> in the plane have as their shared fundamental invariants of a pair of points (u,v) the <u>distance</u> $\|u-v\|$ and the <u>oriented angle</u> $\theta - \psi$. It is natural to ask: what is the proper mathematical "home" for oriented angles? The answer is obvious once you see it: first we identify u, v with complex numbers:

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{C} \\ u &\longmapsto \|u\| e^{i\theta} =: z_u \\ v &\longmapsto \|v\| e^{i\psi} =: z_v \end{aligned}$$

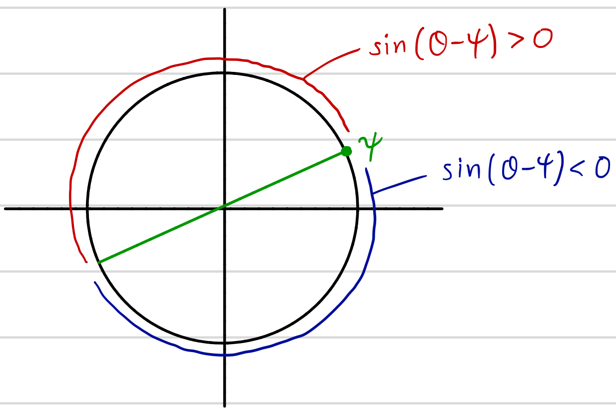
Then we calculate

$$\begin{aligned} z_u \overline{z_v} &= \|u\| e^{i\theta} \cdot \|v\| e^{-i\psi} \\ &= \|u\| \|v\| \, e^{i(\theta-\psi)} . \end{aligned}$$

$\longleftarrow$ records $\theta - \psi \in \mathbb{R}/2\pi\mathbb{Z}$

$$\mathrm{Re}(z_u \overline{z_v}) = \|u\| \cdot \|v\| \cdot \cos(\theta - \psi) = \langle u, v \rangle .$$

The extra information in $z_u \overline{z_v}$ is precisely $\sin(\theta-\psi)$. But once you know $\cos(\theta-\psi)$, and therefore $\sin(\theta-\psi)^2 = 1 - \cos(\theta-\psi)^2$, knowing the actual <u>value</u> of $\sin(\theta-\psi)$ is just the information of the <u>sign</u> of $\sin(\theta-\psi)$, which tells you which hemisphere of S^1 the angle θ lies in, when we bisect the circle at ψ, i.e.



<u>Upshot</u> $\arg(z_u \overline{z_v}) \in \mathbb{R}/2\pi\mathbb{Z}$ is the equivalence class of the oriented angle measured counterclockwise <u>from</u> v to u.

Given two continuous complex-valued functions $f, g : X \longrightarrow \mathbb{C}$ you should think of the values $f(x), g(x)$ as complex numbers (visualised as arrows in $\mathbb{R}^2$) attached at the point x. Then the interesting information in

$$f(x)\overline{g(x)} = |f(x)||g(x)|\, e^{i(\arg f(x) - \arg g(x))}$$

is the oriented angle from $g(x)$ to $f(x)$, and the pairing in $L^2(X, \mathbb{C})$

$$\langle f, g \rangle = \int_X f \bar{g}$$

accumulates these complex numbers. We can visualise this integral more "geometrically" by imagining the vector addition of all the $f(x)\overline{g(x)}$. We will do this exercise below for $X = S^1$, but for even more pictures of this kind of thing see Feynman's book "QED: the strange theory of light and matter" (what a beautiful book).

<u>Lemma L21-3</u> For $m, n \in \mathbb{Z}$ we have $\langle e^{im\theta}, e^{in\theta} \rangle = 2\pi\, \delta_{m,n}$ in $L^2(S^1, \mathbb{C})$.

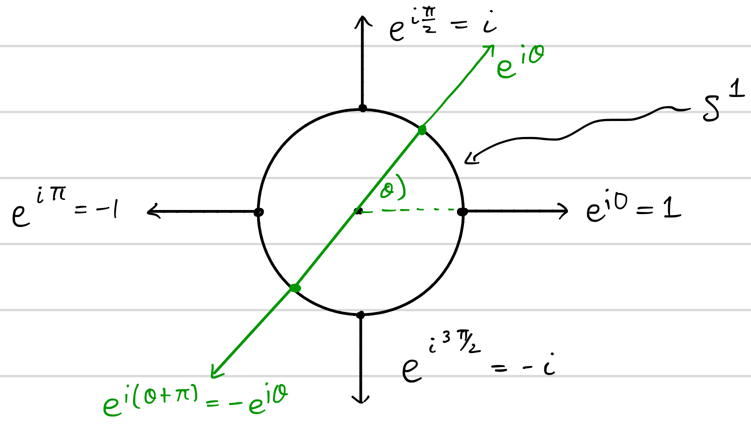
<u>Proof</u> Here $\delta_{m,n} = 1$ if $m = n$ and zero otherwise is called the Kronecker delta. By definition (with $\int = \int_{S^1}$)

$$\langle e^{im\theta}, e^{in\theta} \rangle = \int e^{im\theta} e^{-in\theta} = \int e^{i(m-n)\theta}$$

which is certainly 2π if $m = n$ since $\int 1 = 2\pi$. If $m \neq n$ this is

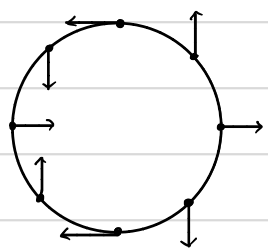
$$\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \left[\frac{-i}{m-n} e^{i(m-n)\theta} \right]_0^{2\pi} = 0. \qquad \square$$

<u>Remark</u> The orthogonality of the $e^{in\theta}$'s is so fundamental that you ought to have a "gut level" understanding of <u>why</u> it is true: consider first $e^{i\theta} : S^1 \longrightarrow \mathbb{C}$ which is just the homeomorphism $S^1 \cong U(1)$ of Tutorials 4, 6.



In the integral $\int_0^{2\pi} e^{i\theta} d\theta$ every contribution $e^{i\theta}$ is precisely cancelled by a $-e^{i\theta}$ corresponding to a phase shift of π, so $\int_0^{2\pi} e^{i\theta} d\theta = 0$.

For $e^{in\theta}$ with $n \neq 0$ the complex numbers attached to points of increasing θ perform n complete periods as θ varies over $[0, 2\pi]$, and hence the integral can be divided as a sum of n parts, each of which is zero by the above calculation: e.g. in the $n=2$ case (not drawing to scale)



<u>Def^n</u> An <u>orthogonal family</u> in an inner product space $(V, \langle , \rangle)$ is an indexed set of vectors $\{u_i\}_{i \in I}$ such that $\langle u_i, u_j \rangle = 0$ if $i \neq j$. An <u>orthonormal family</u> is such an indexed set with $\langle u_i, u_j \rangle = \delta_{ij}$ for all i, j.

If $\{u_i\}_{i\in I}$ is an orthogonal family of nonzero vectors then $\{ u_i/\|u_i\| \}_{i \in I}$ is an orthonormal family.

Example L21-1 The family $\{\frac{1}{\sqrt{2\pi}} e^{in\theta}\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2(S^1, \mathbb{C})$.

Set $u_n := \frac{1}{\sqrt{2\pi}} e^{in\theta}$. We know the set $\{u_n\}_{n\in\mathbb{Z}}$ spans a dense subset of $L^2(S^1,\mathbb{C})$ and our goal since Lecture 17 has been, given $f: S^1 \longrightarrow \mathbb{C}$ continuous (or as we have more recently learned, we could hope to do the same for any integrable function) to find an algorithm for producing a sequence $(p_m)_{m=0}^{\infty}$ in the span of the u_n's with $p_m \longrightarrow f$ as $m \to \infty$ in $L^2(S^1, \mathbb{C})$ (to be completely honest our original goal was to do this for $f: S^1 \longrightarrow \mathbb{R}$, but we can apply the constuction in the complex case and then take real parts). However to make sense of this we need to define what we mean by an infinite series

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} \in L^2(S^1, \mathbb{C}) \qquad a_n \in \mathbb{C}.$$

Lemma L21-4 If $\{u_i\}_{i=1}^n$ is a finite orthogonal family in an inner-product space then

$$\left\| \sum_{i=1}^n u_i \right\|^2 = \sum_{i=1}^n \|u_i\|^2$$

Proof $\|\sum_i u_i\|^2 = \langle \sum_i u_i, \sum_j u_j \rangle = \sum_{i,j} \langle u_i, u_j\rangle = \sum_i \langle u_i, u_i\rangle = \sum_i \|u_i\|^2$. $\square$

Most of the following material is from Cheney "Analysis for applied mathematics" §2.2.

Lemma L21-5 (General Pythagorean law) Let $\mathcal{C} = \{u_i\}_{i\geq 0}$ be an orthogonal family in a Hilbert space H. The series $\sum_{i=0}^{\infty} u_i$ converges if and only if $\sum_{i=0}^{\infty} \|u_i\|^2 < \infty$. If $\sum_{i=0}^{\infty} \|u_i\|^2 = \lambda < \infty$ then $\|\sum_{i=0}^{\infty} u_i\|^2 = \lambda$ and the sum $\sum_i u_i$ is independent of the ordering of the terms.

<u>Proof</u> Set $S_n = \sum_{i=0}^n u_i$ and $s_n = \sum_{i=0}^n \|u_i\|^2$ so that convergence of the series $\sum_{i=0}^\infty u_i$ means $(S_n)_{n=0}^\infty$ converges (or equivalently, is Cauchy).

For $m > n$ we have

$$\| S_m - S_n \|^2 = \| \sum_{j=n+1}^m u_j \|^2 = \sum_{j=n+1}^m \| u_j \|^2 = | s_m - s_n |$$

so $(S_n)_{n=0}^\infty$ is Cauchy in H iff. $(s_n)_{n=0}^\infty$ is Cauchy in $\mathbb{R}$, which proves the first claim (we have used H is complete here already). Now assume that $\lambda < \infty$. By the Pythagorean law $\|S_n\|^2 = s_n$ and hence in the limit

$$\| \sum_{i=0}^\infty u_i \|^2 = \lim_{n\to\infty} \| S_n \|^2 = \lim_{n\to\infty} s_n = \lambda.$$

It remains to prove the claim about the unordered sum. Let us first consider a bijection $f: \mathbb{N} \longrightarrow \mathbb{N}$ and the series $\sum_{i=0}^\infty u_{f(i)}$. Set $U_n = \sum_{i=0}^n u_{f(i)}$. By the theory of absolutely convergent series in $\mathbb{R}$, $\sum_{i=0}^\infty \| u_{f(i)} \|^2 = \lambda$ and so by what we have already said U_n converges, say to $u \in H$, and $\|u\|^2 = \lambda$. Now we compute

$$\langle U_n, S_m \rangle = \langle \sum_{i=0}^n u_{f(i)}, \sum_{j=0}^m u_j \rangle = \sum_{i=0}^n \sum_{j=0}^m \|u_j\|^2 \delta_{j\, f(i)}$$

The pairing $\langle -, S_m \rangle : H \longrightarrow \mathbb{F}$ is continuous by Ex. L20-5 so we compute

$$\langle u, S_m \rangle = \lim_{n\to\infty} \langle U_n, S_m \rangle = \lim_{n\to\infty} \underbrace{\underbrace{\sum_{i=0}^n \sum_{j=0}^m \|u_j\|^2 \delta_{j\, f(i)}}_{\text{equals } \|u_{f(i)}\|^2 \text{ if } f(i) \in \{0,\dots,m\} \text{ and zero otherwise}}}_{\text{equals } \sum_{\substack{i \le n \\ f(i) \le m}} \| u_{f(i)} \|^2}$$

$$= \sum_{\substack{i \geq 0 \\ f(i) \leq m}} \| u_{f(i)} \|^2 = \sum_{j \leq m} \| u_j \|^2$$

Using continuity in the other variable we find $\langle u, \sum_{i=0}^{\infty} u_i \rangle = \sum_{i=0}^{\infty} \|u_i\|^2 = \lambda$. Hence $u = \sum_{i=0}^{\infty} u_i$ since

$$\| u - \sum_{i=0}^{\infty} u_i \| = \|u\|^2 - 2\operatorname{Re}\left(\langle u, \sum_{i=0}^{\infty} u_i \rangle\right) + \lambda$$
$$= \lambda - 2\lambda + \lambda = 0.$$

This shows that any permutation of $\{u_i\}_{i=0}^{\infty}$ also sums to $\sum_{i=0}^{\infty} u_i = \lim_{n \to \infty} S_n$. $\square$

<u>Lemma L21-6</u> Let $\{u_i\}_{i=1}^{n}$ be an orthonormal family in an inner-product space V and ret $U = \operatorname{span}_{\mathbb{F}}(\{u_i\})$. Then for $v \in V$ the closest point to v in U is $\sum_{i=1}^{n} \langle v, u_i \rangle u_i$.

<u>Proof</u> Set $P(v) := \sum_{i=1}^{n} \langle v, u_i \rangle u_i \in U$. By Lemma L20-6 it suffices to show $\langle v - P(v), y \rangle = 0$ for all $y \in U$. But since the pairing is linear it is enough to check this for all $y = u_j$. Hence

$$\langle v - P(v), u_j \rangle = \langle v, u_j \rangle - \sum_{i=1}^{n} \langle v, u_i \rangle \langle u_i, u_j \rangle$$

$$= \langle v, u_j \rangle - \langle v, u_j \rangle = 0$$

proves the claim. $\square$

<u>Lemma L21-7</u> In the notation of the previous lemma, if $v \in U$ then $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$.

The vector $P(v)$ in the above proof is called the <u>orthogonal projection</u> onto U (note that the description as the closest vector in U shows it is independent of which orthonormal spanning set for U we choose). The scalars $\langle v, u_i \rangle$ are called <u>(generalised) Fourier coefficients</u>, for a reason we will explain shortly.

<u>Theorem L21-8</u> (Bessel's inequality) Let $\{u_i\}_{i\in I}$ be an orthonormal system in an inner-product space V with I countable. Then for $v \in V$

$$\sum_{i\in I} |\langle v, u_i \rangle|^2 \leq \|v\|^2.$$

<u>Proof</u> If I is finite we know what the LHS means. If I is infinite, choose an ordering $I = \{i_0, i_1, \ldots\}$. We show $\sum_{k=0}^{\infty} |\langle v, u_{i_k} \rangle|^2$ converges, hence converges absolutely, so any other ordering of I produces a convergent series with the same limit and we might as well assume $I = \{0, 1, \ldots\}$ to begin with. Given $v \in V$ we write v_n for $\sum_{i=0}^{n} \langle v, u_i \rangle u_i$. By Lemma L21-6 we have $v - v_n$ orthogonal to $U_n = \operatorname{span}_{\mathbb{F}}(\{u_i\}_{i=0}^{n})$. Hence by the Pythagorean law

$$\begin{aligned} \|v\|^2 = \|v - v_n + v_n\|^2 &= \|v - v_n\|^2 + \|v_n\|^2 \\ &\geq \|v_n\|^2 \\ &= \Big\| \sum_{i=0}^{n} \langle v, u_i \rangle u_i \Big\|^2 \\ &= \sum_{i=0}^{n} |\langle v, u_i \rangle|^2 \end{aligned}$$

Taking the $n \to \infty$ limit proves the claim (the sum is positive and bounded above, hence converges). $\square$

<u>Corollary L21-9</u> If $\{u_i\}_{i=0}^{\infty}$ is an orthonormal system in an inner product space and $v \in V$

$$\lim_{n\to\infty} \langle v, u_n \rangle = 0.$$

<u>Proof</u> By Bessel's inequality $\lim_{n\to\infty}\sum_{i=0}^{n}|\langle v,u_i\rangle|^2<\infty$ so this is immediate. □

<u>Exercise L21-1</u> Let $\{u_i\}_{i\in I}$ be an orthogonal family of nonzero vectors in an inner-product space V. Then this set is linearly independent. Use this to give an independent proof of Lemma L21-1 (although as a matter of taste, the old proof is more elementary and thus "better").

<u>Def^n</u> A <u>countable orthonormal dense basis</u> (hence <u>dense basis</u>) in an inner-product space V is an orthonormal family $\{u_i\}_{i\in I}$ with I countable such that

$$V = \overline{\operatorname{span}_{\mathbb{F}}(\{u_i\}_{i\in I})}$$

uncountable dense bases are also very important, but we do not have time to develop the theory in that generality!

<u>Example L21-2</u> $\left\{\frac{1}{\sqrt{2\pi}}e^{in\theta}\right\}_{n\in\mathbb{Z}}$ is a dense basis for $L^2(S^1,\mathbb{C})$ by Lemma L21-2, L21-3.

<u>Exercise L21-2</u> Let V be a topological vector space over $\mathbb{F}$ and $U\subseteq V$ a vector subspace. Prove that $\overline{U}$, the closure of U, is also a vector subspace.

A set is called <u>countable</u> if it is bijective to some subset of $\mathbb{N}$ (so finite sets are countable).

<u>Theorem L21-10</u> For a countable orthonormal family $\{u_i\}_{i\in I}$ in a Hilbert space H, the following are equivalent:

(i) $\{u_i\}_{i\in I}$ is a dense basis.
(ii) If $h\in H$ and $\langle h,u_i\rangle=0$ for all $i\in I$ then $h=0$.
(iii) If $h\in H$ then $h=\sum_{i\in I}\langle h,u_i\rangle u_i$.
(iv) If $h,k\in H$ then $\langle h,k\rangle=\sum_{i\in I}\langle h,u_i\rangle\langle u_i,k\rangle$.
(v) If $h\in H$ then $\|h\|^2=\sum_{i\in I}|\langle h,u_i\rangle|^2$. (Parseval identity)

<u>Proof</u> Part of the proof consists in showing the sums in (iii), (iv), (v) exist and are independent of how we enumerate I. For (i) $\Rightarrow$ (ii) note that if h is orthogonal to each u_i it is orthogonal to $U := \mathrm{span}_{\mathbb{F}}(\{u_i\}_{i\in I})$. But there is a sequence $(p_n)_{n=0}^{\infty}$ in U converging to h, so by continuity of the pairing

$$\langle h, h\rangle = \langle h, \lim_{n\to\infty} p_n\rangle = \lim_{n\to\infty} \langle h, p_n\rangle = 0$$

Hence $h = 0$ as claimed.

(ii) $\Rightarrow$ (iii) Let $h \in H$ be given. To show the sum $h' = \sum_{i\in I} \langle h, u_i\rangle u_i$ converges and is independent of how we order the terms, it suffices by Lemma L21-5 to show that some (and therefore any) enumeration $I = \{i_0, i_1, \dots\}$ that

$$\sum_{k=0}^{\infty} \| \langle h, u_{i_k}\rangle u_{i_k} \|^2 = \sum_{k=0}^{\infty} |\langle h, u_{i_k}\rangle|^2 < \infty$$

But this is immediate from Bessel's inequality (Theorem L21-8). So h' exists, and moreover

$$\begin{aligned}\langle h - h', u_i\rangle &= \langle h, u_i\rangle - \langle h', u_i\rangle \\ &= \langle h, u_i\rangle - \langle \lim_{n\to\infty} \textstyle\sum_{k=0}^{n} \langle h, u_{i_k}\rangle u_{i_k}, u_i\rangle \\ &= \langle h, u_i\rangle - \lim_{n\to\infty} \textstyle\sum_{k=0}^{n} \langle h, u_{i_k}\rangle \langle u_{i_k}, u_i\rangle \\ &= \langle h, u_i\rangle - \langle h, u_i\rangle = 0\end{aligned}$$

So by hypothesis $h - h' = 0$ and thus $h = h'$.

(iii) $\Rightarrow$ (iv) As we have just shown, $\sum_{i\in I} \langle h, u_i\rangle u_i$ is unconditionally convergent (i.e. any enumeration of I leads to a series converging to the same limit). Then

$$\langle h,k\rangle = \langle \lim_{n\to\infty} \sum_{k=0}^{n} \langle h,u_{i_k}\rangle u_{i_k}, k\rangle$$
$$= \lim_{n\to\infty} \sum_{k=0}^{n} \langle h,u_{i_k}\rangle\langle u_{i_k},k\rangle$$

This shows that $\sum_{i\in I}\langle h,u_i\rangle\langle u_i,k\rangle$ also converges unconditionally. Actually, this unconditional converge did not require (iii)!

(iv) $\Rightarrow$ (v) We have just established $\sum_{i\in I}\langle h,u_i\rangle\langle u_i,k\rangle$ converges unconditionally. We now suppose it is always equal to $\langle h,k\rangle$. Putting $h=k$ yields

$$\|h\|^2 = \sum_{i\in I}\langle h,u_i\rangle\overline{\langle h,u_i\rangle} = \sum_{i\in I}|\langle h,u_i\rangle|^2.$$

(v) $\Rightarrow$ (i) Let U be as above. By Ex. L21-2, $\overline{U}$ is a (closed) vector subspace of H (see also Ex. L18-10) and so by Lemma L20-7

$$H = \overline{U}\oplus\overline{U}^{\perp}.$$

We suppose for a contradiction that $\overline{U}\neq H$, so there is a nonzero $v\in\overline{U}^{\perp}$. We may assume $\|v\|=1$. But then by (v),

$$1 = \|v\|^2 = \sum_{i\in I}|\langle v,u_i\rangle|^2 = 0$$

which is a contradiction. Hence $\overline{U}=H$. $\square$

<u>Remark</u> What we have called an orthonormal dense basis is sometimes just called an <u>orthonormal basis</u>. However this is problematic because such a thing is <u>not</u> a basis which happens to be an orthonormal set. I have chosen the appellation "dense" since that seems informative, but be aware this is <u>not</u> standard terminology (orthonormal basis, or Schauder basis, is).

<u>Example L21-3</u> Set $u_n = \frac{1}{\sqrt{2\pi}} e^{in\theta}$. The set $\{u_n\}_{n\in\mathbb{Z}}$ is a (countable) orthonormal dense basis for $L^2(S^1, \mathbb{C})$, and so we deduce from the theorem that for any $f \in L^2(S^1, \mathbb{C})$

$$f = \sum_{n=-\infty}^{\infty} \langle f, u_n \rangle u_n .$$

Moreover the convergence of this series is unconditional, so we can enumerate $\mathbb{Z}$ in any way we like, for example

$$p_N = \sum_{n=-N}^{N} \langle f, u_n \rangle u_n \longrightarrow f \quad \text{as } N \to \infty.$$

This achieves our long standing goal of finding an algorithm for computing a sequence of trigonometric polynomials converging to f. Note that the coefficients are, if $f: S^1 \to \mathbb{C}$ is continuous, integrals

$$\begin{aligned} \langle f, u_n \rangle &= \int_{S^1} f \overline{u_n} \\ &= \frac{1}{\sqrt{2\pi}} \int_{S^1} f e^{-in\theta} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta \qquad \text{(Riemann integral)} \end{aligned}$$

These complex numbers are called the <u>Fourier components</u> of f. The linear functionals $\langle -, u_n \rangle : L^2(S^1, \mathbb{C}) \longrightarrow \mathbb{C}$ "read off" how much of f is in the "direction" u_n, so there is a precise analogy with the dual basis in a finite-dimensional vector space (the conceptual message here is that what <u>actually</u> made finite-dimensional spaces great was that they were <u>self-dual</u>. Once we know how to build self-dual infinite-dimensional spaces, i.e. Hilbert spaces, many things generalise).

The story of trigonometric polynomials is not quite complete, because originally we were concerned with real-valued functions, and real-valued trigonometric polynomials. We can get ourselves back to this context by "taking real parts" in the following sense.

Recall that given a pair of vector spaces A, B the (external) direct sum $A \oplus B$ is the set $A \times B$ with $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$, $\lambda(a_1, b_1) = (\lambda a_1, \lambda b_1)$. There is an isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Cts}(X, \mathbb{C}) \xrightarrow{\ \phi\ } \mathrm{Cts}(X, \mathbb{R}) \oplus \mathrm{Cts}(X, \mathbb{R})$$

$$\phi(f) = (\mathrm{Re}(f), \mathrm{Im}(f))$$

where i acts as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the right hand side. Suppose now that $(X, \int_X)$ is an integral pair and we take the associated $\|-\|_2$-norm on $\mathrm{Cts}(X, \mathbb{C})$, $\mathrm{Cts}(X, \mathbb{R})$. The right-hand side of the above becomes a normed space over $\mathbb{C}$ with the norm

$$\|(f_1, f_2)\|_2 := \|f_1\|_2 + \|f_2\|_2$$

and with this structure ϕ is an isomorphism of normed spaces over $\mathbb{C}$.

Lemma L21-11 There is an isomorphism of normed spaces over $\mathbb{C}$

$$\Phi : L^2(X, \mathbb{C}) \xrightarrow{\ \cong\ } L^2(X, \mathbb{R}) \oplus L^2(X, \mathbb{R})$$

where i acts as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the right, making the following diagram commute

$$\begin{tikzcd}
L^2(X, \mathbb{C}) \arrow[r, "\Phi"] & L^2(X, \mathbb{R}) \oplus L^2(X, \mathbb{R}) \\
\mathrm{Cts}(X, \mathbb{C}) \arrow[u, "\iota"] \arrow[r, "\phi"'] & \mathrm{Cts}(X, \mathbb{R}) \oplus \mathrm{Cts}(X, \mathbb{R}) \arrow[u, "\iota \oplus \iota"']
\end{tikzcd}$$

Proof It is immediate from Theorem L18-9 that if two normed spaces are isomorphic their completions are isomorphic. Then using Ex. L18-12 (which is clearly also an isomorphism of normed spaces $(V \oplus W)^\wedge \cong V^\wedge \oplus W^\wedge$ if V, W are normed spaces) we have as normed spaces over $\mathbb{C}$

$$\begin{aligned} L^2(X, \mathbb{C}) &= (C_5(X, \mathbb{C}), \|-\|_2)^\wedge \\ &\cong \{ (C_5(X, \mathbb{R}), \|-\|_2) \oplus (C_5(X, \mathbb{R}), \|-\|_2) \}^\wedge \\ &\cong (C_5(X, \mathbb{R}), \|-\|_2)^\wedge \oplus (C_5(X, \mathbb{R}), \|-\|_2)^\wedge \\ &= L^2(X, \mathbb{R}) \oplus L^2(X, \mathbb{R}). \end{aligned}$$

Here all we really need to check is that the action of i matches up, which it does. $\square$

Given $f \in L^2(S^1, \mathbb{C})$ we write $Re(f)$, $Im(f)$ for $\Phi(f)_1$, $\Phi(f)_2$ resp.

Example L21-4 So $\Phi : L^2(S^1, \mathbb{C}) \xrightarrow{\cong} L^2(S^1, \mathbb{R}) \oplus L^2(S^1, \mathbb{R})$ is continuous, and if $f : S^1 \longrightarrow \mathbb{R}$ is continuous then we may first of all view it as an element of $L^2(S^1, \mathbb{C})$ (via $\mathbb{R} \subseteq \mathbb{C}$) and then in $L^2(S^1, \mathbb{R})$:

$$\begin{aligned} f &= Re(f) \\ &= Re\left(\lim_{N \to \infty} \sum_{n=-N}^{N} \langle f, u_n \rangle u_n \right) \\ &= \lim_{N \to \infty} \sum_{n=-N}^{N} Re\left(\langle f, u_n \rangle u_n \right) \end{aligned}$$

Suppose $\int_0^{2\pi} f(\theta) e^{-in\theta} d\theta = a_n + i b_n$ with $a_n, b_n \in \mathbb{R}$. Then

$$\begin{aligned} Re(\langle f, u_n \rangle u_n) &= Re\left[\tfrac{1}{2\pi} (a_n + i b_n)(\cos(n\theta) + i \sin(n\theta)) \right] \\ &= \tfrac{1}{2\pi} (a_n \cos(n\theta) - b_n \sin(n\theta)) \end{aligned}$$

Hence as a limit in $L^2(S^1,\mathbb{R})$

$$f = \lim_{N\to\infty} \sum_{n=-N}^{N} \tfrac{1}{2\pi}\big(a_n\cos(n\theta) - b_n \sin(n\theta)\big)$$

$$= \lim_{N\to\infty} \tfrac{1}{2\pi}\Big(a_0 + \sum_{n=1}^{\infty} [a_n + a_{-n}]\cos(n\theta) + \sum_{n=1}^{\infty} [-b_n + b_{-n}]\sin(n\theta)\Big) \qquad (18.1)$$

which is the desired expression of f as a limit of vectors in $\mathrm{TPoly}(S^1,\mathbb{R})$.

<u>Exercise L21-3</u> Prove that $\{1\}\cup\{\cos(n\theta),\sin(n\theta)\}_{n>0}$ is an orthogonal family in $L^2(S^1,\mathbb{R})$. Derive from this an orthonormal dense basis and thereby give an independent derivation of (18.1).

We can now claim to understand complex and real-valued functions on the circle. But if we are honest about it the answer is deeply surprising: the natural <u>coordinates</u> on $L^2(S^1,\mathbb{C})$ (i.e. the functions $\langle -, u_n\rangle$), or if you like the fundamental degrees of freedom in a vector $f\in L^2(S^1,\mathbb{C})$, are <u>non-local</u> with respect to the original space S^1. The natural "directions" in $L^2(S^1,\mathbb{C})$ from f consist of variations in the amplitude of individual frequency components $e^{in\theta}$ of f, and such a variation <u>changes the value of f at every point of S^1</u>. We say the change is "local" in <u>frequency space</u> (physicists will often call this momentum space) but "non-local" in <u>position space</u>, meaning S^1 itself. In fact, there is a precise sense in which $L^2(S^1,\mathbb{C})$ does not "believe" in points at all.

This relation between position space S^1 and frequency space $\mathbb{Z}$ is actually an instance of a general duality, called <u>Pontryagin duality</u>, on locally compact topological abelian groups: many of the L^2-spaces appearing as the Hilbert spaces of physics or applied mathematics have the property that the natural coordinate directions represent non-local perturbations, of a wave-like nature.

We have focused on the case of S^1 for concreteness, but most of the above works more generally: for the following exercises let $(X, \int_X)$ be an integral pair and $j: X \longrightarrow \mathbb{R}^n$ an embedding, and let $z_1, \ldots, z_n \in \mathrm{Cts}(X, \mathbb{R})$ denote the maps $z_i := \pi_i \circ j$ where the π_i are the projections, so

$$\mathcal{C} = \{ z_1^{a_1} \cdots z_n^{a_n} \mid a_1, \ldots, a_n \geq 0 \} \subseteq \mathrm{Cts}(X, \mathbb{R})$$

is a countable set spanning (by Stone-Weierstrass, see Corollary L16-4) a vector subspace $\mathrm{span}_{\mathbb{R}} \mathcal{C}$ which is dense in $\mathrm{Cts}(X, \mathbb{R})$ (with respect to $\|-\|_\infty$, and thus also $\|-\|_2$) and so $\mathrm{span}_{\mathbb{R}} \mathcal{C}$ is also dense in $L^2(X, \mathbb{R})$, whence by Lemma L21-11 the subspace $\mathrm{span}_{\mathbb{C}} \mathcal{C}$ is also dense in $L^2(X, \mathbb{C})$.

Exercise L21-4 Prove that $\mathcal{C}$ contains a subset $\mathcal{C}'$ which is linearly independent and spans the same vector subspace.

Exercise L21-5 Invent a generalised form of the Gram-Schmidt process which takes an enumeration $\mathcal{C}' = \{c_0, c_1, \ldots\}$ and produces an orthonormal family $\{u_n\}_{n=0}^\infty$ spanning the same vector subspace as $\mathcal{C}$ (in particular the u_n are polynomials in the z_i). Conclude that $\{u_n\}_{n=0}^\infty$ is an orthonormal dense basis (in either $L^2(X, \mathbb{R})$ or $L^2(X, \mathbb{C})$, the same set works in both) consisting of polynomial functions.

For example, with $X = [-1, 1]$ and $\mathcal{C} = \{ z^n \mid n \geq 0 \}$ this produces as an orthonormal dense basis for $L^2([-1,1], \mathbb{R})$, $L^2([-1,1], \mathbb{C})$: the (normalised) Legendre polynomials. You might enjoy investigating $L^2(S^1 \times S^1, \mathbb{C})$ or $L^2(S^2, \mathbb{C})$, in the former case you should discover a description in terms of two independent frequencies, and in the latter case you should discover spherical harmonics. What is L^2 of a graph?