# Lecture 22: Urysohn's Lemma

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An important theme in this course has been the study of functions $f: X \longrightarrow \mathbb{R}$ or $f: X \longrightarrow \mathbb{C}$ on a space $X$ via approximations of general functions by "simpler" ones. For example, we saw in Lecture 16 how to approximate continuous functions on $[a,b]$ by polynomial functions (the Weierstrass approximation theorem) and in Lecture 20 how to approximate integrable functions by continuous ones (using the Riesz representation theorem). Indeed, as we explained in Lecture 15 in the context of Picard's theorem on ODEs, the fundamental importance of such approximations explains why so much of the course was organised around putting topologies on spaces of functions (first the compact-open topology, and then the topology associated to the $L^p$-norms) since it is these topologies which give meaning to the term "approximation".

Our most powerful tool for producing approximations is the Stone-Weierstrass theorem of Lecture 16 (in Lecture 21 we even saw how to combine this with the theory of Hilbert spaces to make the approximations effectively computable in terms of integrals). Recall:

Theorem L16-3 (Stone-Weierstrass) Let $X$ be a compact Hausdorff space and $A \subseteq \mathrm{Cts}(X,\mathbb{R})$ a subalgebra which separates points. Then we have $\overline{A} = \mathrm{Cts}(X,\mathbb{R})$. ← compact-open topology

⌜Recall a subset $A$ separates points if $\forall x,y \in X (x \neq y \Rightarrow \exists f \in A(fx \neq fy))$⌟

If $j: X \longrightarrow \mathbb{R}^n$ is an embedding then the restriction of polynomial functions $\mathbb{R}^n \longrightarrow \mathbb{R}$ gives a subalgebra of $\mathrm{Cts}(X,\mathbb{R})$ which is easily seen to separate points, and is therefore dense, by Stone-Weierstrass. This has been our primary source of approximations of continuous functions by "simple" continuous functions; see for example Exercise L21-4, L21-5.

Let us see what happens when we take away the "crutch" of having $X$ embedded in $\mathbb{R}^n$.

We fall flat on our face is what happens: for a general topological space $X$, we don't know a single interesting continuous function $X \longrightarrow \mathbb{R}$! Let alone a collection of such functions rich enough to approximate all the other ones.

So, what continuous functions $f: X \longrightarrow \mathbb{R}$ do we know?

- Constant functions (i.e. not interesting functions)
- If $X$ is metrisable, with metric $d$, then for any $B \subseteq X$ the function $d(-, B): X \longrightarrow \mathbb{R}$ is continuous (Lemma L13-3).

That isn't exactly an impressive list. Although, if $X$ has the indiscrete topology $\mathcal{T} = \{\phi, X\}$ every continuous function $f: X \longrightarrow \mathbb{R}$ is constant, which explains why we don't know anything "generically" about interesting continuous functions: in general there aren't any! But under some reasonable hypotheses, say compact Hausdorff or more generally normality of $X$ (see Ex. L11-8 "metrisable implies normal" and Ex. L11-9 "compact Hausdorff implies normal") we could hope to do better.

Exercise L22-1 Prove that if $(X, d)$ is a metric space and $B \subseteq X$ is dense, then $\{d(-, b)\}_{b \in B}$ is a collection of continuous functions which separates points (and therefore generates a dense subalgebra of $\mathrm{Cts}(X, \mathbb{R})$ provided $X$ is compact).

Indeed, the Urysohn Lemma and its corollary, the Tietze extension theorem, provide powerful tools for constructing continuous functions $X \longrightarrow \mathbb{R}$ with some specified behaviour, for any normal space $X$. They are among the most widely used tools in topology.

The proof we give below is (mostly) following Munkres.

<u>Lemma L22-1</u> Let $X$ be a topological space in which points are closed. Then $X$ is normal if and only if for every pair $A \subseteq B \subseteq X$ with $A$ closed and $B$ open, there exists an open set $U$ with $A \subseteq U \subseteq \overline{U} \subseteq B$. ⌜$\overline{U}$ is the closure in $X$⌟

<u>Proof</u> Assume $X$ is normal and that $A \subseteq B$ is given. Then $B^c = X \setminus B$ is closed, and so by normality there exist open disjoint sets $U, V$ with $A \subseteq U$, $B^c \subseteq V$. Hence $U \subseteq V^c \subseteq B$ and since $V^c$ is closed, $\overline{U} \subseteq V^c \subseteq B$ also, so we have $A \subseteq U \subseteq \overline{U} \subseteq B$.

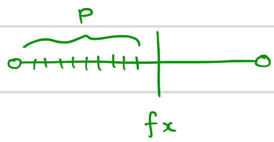
Conversely, suppose the condition on the existence of $U$ holds, and let $A, B$ be disjoint closed subsets of $X$. Then $A \subseteq B^c \subseteq X$ so there exists $U$ open with $A \subseteq U \subseteq \overline{U} \subseteq B^c$. But then $U, \overline{U}^c$ are disjoint open sets with $A \subseteq U$ and $B \subseteq \overline{U}^c$ so $X$ is normal. □

<u>Theorem L22-2</u> (Urysohn's Lemma) Let $A, B$ be disjoint closed subsets of a normal space $X$. There exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

<u>Proof</u> Suppose we had such an $f$. We want to examine what this tells us about the topology of $X$ (it certainly tells us something: as long as $A, B$ are nonempty $f$ is not constant, so $X$ cannot have the indiscrete topology). The set $[0,p) \subseteq [0,1]$ is open for $0 < p < 1$ and hence $U_p := f^{-1}([0,p))$ is open. Notice that $A \subseteq U_p \subseteq B^c$ for all $p$, and if $0 < p < q < 1$ then

$$U_p \subseteq f^{-1}[0,p] \subseteq f^{-1}[0,q) = U_q \quad \Rightarrow \quad \overline{U_p} \subseteq U_q.$$

The key insight is that $f$ can be recovered from the system of open sets $\{U_p\}_{0<p<1}$ !
And in fact we only need a dense subset of the indices: suppose $P \subseteq (0,1)$ is dense, then for $x \in X$ we have




$$
\begin{aligned}
fx &= \sup\left( \{ p \in P \mid fx \geqslant p \} \cup \{0\} \right) \\
&= \sup\left( \{ p \in P \mid fx \notin [0,p) \} \cup \{0\} \right) \qquad (4.1) \\
&= \sup\left( \{ p \in P \mid x \notin U_p \} \cup \{0\} \right).
\end{aligned}
$$

Now let us begin the proof proper, that is, we drop the hypothesis about the existence of $f$. We proceed in two steps: the first step is to show that given a system of open sets $\{U_p\}_{p \in P}$, with $P \subseteq (0,1)$ dense, satisfying the properties discussed above (the existence of such a system of open sets is a hypothesis on the "richness" of the topology of $X$) that (4.1) actually defines a continuous function. The second step is to explain how to produce such a system, using that $X$ is normal.

Claim (Step 1) Suppose $P \subseteq (0,1)$ is dense, and that $\{U_p\}_{p \in P}$ is a family of open subsets of $X$ satisfying

(i) $A \subseteq U_p \subseteq B^c$ for all $p \in P$.

(ii) $\overline{U_p} \subseteq U_q$ for all $p < q$ (closure in $X$)

Then there is a continuous function $f : X \longrightarrow [0,1]$ with $A \subseteq f^{-1}(0)$, $B \subseteq f^{-1}(1)$ defined by

$$
fx = \sup\left( \{ p \in P \mid x \notin U_p \} \cup \{0\} \right).
$$

<u>Proof of claim</u> The given supremum exists and lies in $[0,1]$, and hypothesis (i) ensures that for $a \in A$ we have $fa = \sup(\{0\}) = 0$ and for $b \in B$

$$fb = \sup(P \cup \{0\}) = 1.$$

So we need only prove continuity. Note that by definition, for $p \in P$

$$x \notin U_p \implies fx \geqslant p$$
$$fx < p \implies x \in U_p \quad \text{(this is the contrapositive)}$$

Given $0 < r_0 < s_0 < 1$ with $r_0, s_0 \in P$ and a point $x \in f^{-1}((r_0, s_0))$ then we have

$$r_0 < r < fx < s < s_0 \quad \text{for some } r, s \in P$$

by density. This immediately implies $x \in U_s$. We claim $x \notin \overline{U_r}$. For if $x \in \overline{U_r}$ were true then for any $r' \in P$ with $r < r' \leqslant fx$, $x \in \overline{U_r}$ would imply by (ii) that $x \in \overline{U_r} \subseteq U_{r'}$. But since $r < fx$ there must exist at least one $r' \in P$ with $r < r' \leqslant fx$ and $x \notin U_{r'}$ (since $fx$ is the supremum of such $r'$'s). Hence we may conclude $x \notin \overline{U_r}$ and so $x \in U_s \cap \overline{U_r}^c$. We claim

$$U_s \cap \overline{U_r}^c \subseteq f^{-1}((r_0, s_0)) \qquad (*)$$

which will be enough, since $x$ is generic, to prove that $f^{-1}((r_0, s_0))$ is open. Suppose $y \in U_s \cap \overline{U_r}^c$. Then $fy \geqslant r > r_0$ since $y \notin U_r$ and $fy \leqslant s < s_0$ since $fy = \sup\{p \in P \mid y \notin U_p\}$ and if $y \notin U_p$ then $p \leqslant s$ (since if $s < p$ then by (i) we would have $y \in U_s \subseteq \overline{U_s} \subseteq U_p$). This proves $(*)$ and hence that $f^{-1}((r_0, s_0))$ is open. Similarly $f^{-1}([0, s_0))$ and $f^{-1}((r_0, 1])$ are open, and since $P$ is dense this suffices to show $f$ is continuous. $\square$ end of Claim 1.

<u>Claim</u> (Step 2) Such a system of open subsets $\{U_p\}_{p\in P}$ exists for

$$P = \mathbb{Q} \cap (0,1).$$

<u>Proof of claim</u> The essential point here is that $P$ is <u>countable</u>. Let $P$ be enumerated in some way $P = \{p_1, p_2, \dots\}$ and let $P_n = \{p_1, \dots, p_n\}$ be the first $n$ rational numbers in this enumeration. To define $U_{p_1}$ apply Lemma L22-1 (using that $X$ is normal) to find $U_{p_1}$ open with $A \subseteq U_{p_1} \subseteq \overline{U_{p_1}} \subseteq B^c$. Now suppose we have constructed $U_{p_1}, \dots, U_{p_n}$ in such a way that $\{U_p\}_{p\in P_n}$ satisfies the conditions (i), (ii) from Step 1, and we wish to define $U_{p_{n+1}}$. Set

$$r = \sup\{ p_i \mid 1 \le i \le n \text{ and } p_i < p_{n+1} \} \quad \text{- immediate predecessor}$$
$$s = \inf\{ p_i \mid 1 \le i \le n \text{ and } p_{n+1} < p_i \} \quad \text{- immediate successor}$$

Then $r<s$ so by hypothesis $\overline{U_r} \subseteq U_s$ and by Lemma L22-1 there exists an open set $V$ with $\overline{U_r} \subseteq V \subseteq \overline{V} \subseteq U_s$. We set $U_{p_{n+1}} = V$. Here if $r$ is the supremum of the empty set we read $\overline{U_r}$ as $A$, and if $s$ is the infimum of the empty set we read $U_s$ as $B^c$ (they are not both empty).
We claim $\{U_p\}_{p\in P_{n+1}}$ still satisfies the conditions (i), (ii). The first is clear.

For the second condition, suppose $p_{n+1} < p_i$ for some $i \le n$. Then $s \le p_i$ so by construction

$$\overline{U}_{p_{n+1}} = \overline{V} \subseteq U_s \subseteq \overline{U_s} \subseteq U_{p_i}$$

and similarly if $p_i < p_{n+1}$ then $p_i \le r$ so

$$\overline{U}_{p_i} \subseteq \overline{U}_r \subseteq V = U_{p_{n+1}}.$$

By the principle of recursive definition (a form of induction, see e.g. Munkres for a precise statement) this suffices to define a system $\{U_p\}_{p\in P}$ with the desired properties. $\square$ end of Claim 2

If we now apply Step 1 to one of the systems $\{U_p\}_{p\in\mathbb{Q}\cap(0,1)}$ whose existence is guaranteed by Step 2, we obtain a continuous function $f: X \longrightarrow [0,1]$ of the desired kind. $\square$

Some applications of the Urysohn lemma:

- Tietze's extension theorem: if $X$ is normal and $A \subseteq X$ is closed, and $f: A \longrightarrow \mathbb{R}$ is continuous and bounded then there exists a bounded continuous function $h: X \longrightarrow \mathbb{R}$ with $h|_A = f$.

- Urysohn metrisation theorem: every normal space with a countable basis is metrisable (!).

- Existence of partitions of unity (meaning partitions of unity): this consequence of Urysohn's lemma is in turn the crucial ingredient in showing that any compact topological m-manifold $X$ (i.e. a Hausdorff space with countable basis such that each point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^m$) admits an embedding $X \overset{j}{\hookrightarrow} \mathbb{R}^N$ for some $N$.

This last example means that, in principle, for any compact topological manifold $X$ equipped as an integral pair $(X, \int_X)$, we can approximate arbitrary continuous functions $X \longrightarrow \mathbb{R}$ by "polynomials" obtained from $j: X \longrightarrow \mathbb{R}^N$, and in turn Ex. L21-4, L21-5 produce from these polynomials an orthonormal basis of $L^2(X, \mathbb{C})$. So our analysis of $L^2$-spaces was actually quite general within compact spaces!