Lecture 3 : Isometries

(updated 30/7/18)

Last lecture we proved that (S^1, d_a) was a metric space, and the proof involved certain special functions

$$R_\theta : S^1 \longrightarrow S^1 \qquad R_\theta \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{(rotation)}$$

$$T : S^1 \longrightarrow S^1 \qquad T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \qquad \text{(reflection)}$$

which had the property that they were <u>distance preserving</u>:

$$d_a(x,y) = d_a(R_\theta x, R_\theta y) \qquad \forall x,y \in S^1 \quad \forall \theta \in \mathbb{R}$$
$$d_a(x,y) = d_a(Tx, Ty) \qquad \forall x,y \in S^1.$$

<u>Def^n</u> If $(X, d_X), (Y, d_Y)$ are metric spaces, a function $f : X \to Y$ is <u>distance preserving</u> if

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X.$$

A distance preserving function which is bijective is called an <u>isometry</u>.

<u>Exercise L3-1</u> Prove that any distance preserving function is injective (so such a f^n is an isometry iff. it is surjective).

<u>Example L3-1</u> The functions $\{R_\theta\}_{\theta \in \mathbb{R}}, T$ are isometries of (S^1, d_a).

<u>Exercise L3-2</u> Prove that the set $\mathrm{Isom}(X, d_X)$ of isometries $f : X \to X$ is a <u>group</u> under composition: the identity function is an isometry, the composite of isometries is an isometry, and the inverse of an isometry is an isometry.

That means we can generate new isometries from old ones, e.g:

$$F = R_{\theta_1} R_{\theta_2} T R_{\theta_3} T R_{\theta_4} \in \mathrm{Isom}(S^1, d_a).$$

This raises two natural questions:

$\boxed{Q1}$ Is this really a <u>new</u> isometry? i.e. is F equal to T or R_θ for some θ?

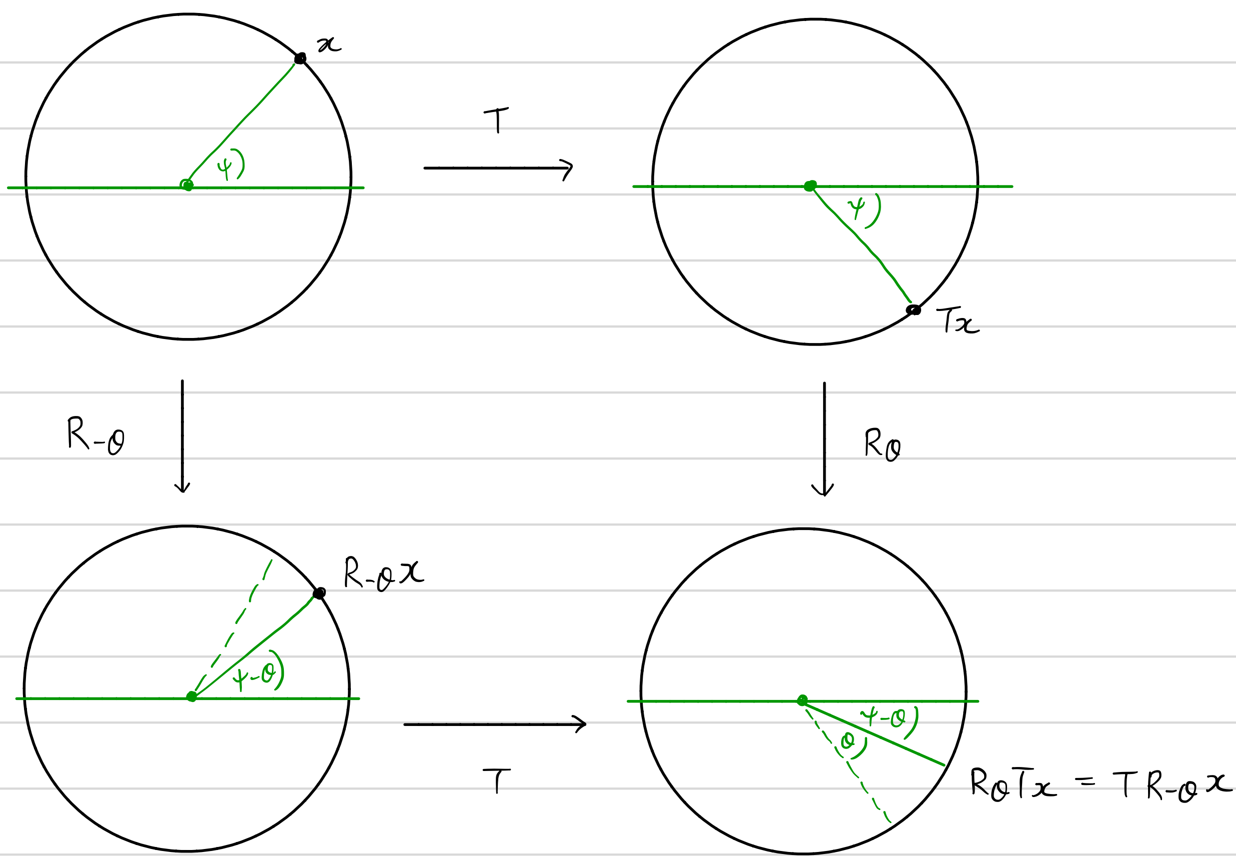
$\boxed{Q2}$ Can we classify <u>all</u> the isometries of (S^1, d_a)? For example, we might hope all isometries can be written as products of R_θ's and T's, like F above. In this case we would say the set $\{R_\theta\}_{\theta \in \mathbb{R}} \cup \{T\}$ <u>generates</u> the group $\mathrm{Isom}(S^1, d_a)$.

Let us begin with Q1. We know that $R_{\theta_1} R_{\theta_2} = R_{\theta_1 + \theta_2}$, so we only really need to analyse the product $R_\theta T$. But for $(a,b) \in S^1$,

$$
\begin{aligned}
R_\theta T \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= T R_{-\theta} \begin{pmatrix} a \\ b \end{pmatrix}.
\end{aligned}
$$

Thus as functions $R_\theta T = T R_{-\theta}$. In other words, the following diagram commutes:

$$
\begin{tikzcd}
S^1 \arrow[r, "T"] \arrow[d, "R_{-\theta}"'] & S^1 \arrow[d, "R_\theta"] \\
S^1 \arrow[r, "T"'] & S^1
\end{tikzcd}
$$



But using the relations

$$
\begin{array}{ll}
(R1) & R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2} \\
(R2) & R_\theta T = T R_{-\theta} \\
(R3) & T^2 = id
\end{array}
$$

in the group of isometries, we may compute that

$$
\begin{aligned}
F &= \underline{R_{\theta_1} R_{\theta_2}}\, T R_{\theta_3} T R_{\theta_4} \\
&= R_{\theta_1+\theta_2} \underline{T R_{\theta_3}}\, T R_{\theta_4} \\
&= R_{\theta_1+\theta_2} R_{-\theta_3} T \underline{T R_{\theta_4}} \\
&= R_{\theta_1+\theta_2} R_{-\theta_3} \underline{T R_{-\theta_4}}\, T \\
&= R_{\theta_1+\theta_2} R_{-\theta_3} R_{\theta_4} T T \\
&= R_{\theta_1+\theta_2-\theta_3+\theta_4} \circ id \\
&= R_\psi
\end{aligned}
$$

where $\psi = \theta_1+\theta_2-\theta_3+\theta_4$.

<u>Exercise L3·3</u> Prove that any element of $\mathrm{Isom}(S^1, d_a)$ of the form

$$F = g_1 \cdots g_r \qquad r \geqslant 0$$

where each g_i is either R_θ for some $\theta \in \mathbb{R}$, or T, may be proven equal to $R_\psi T^n$ for some $\psi \in [0, 2\pi)$ and $n \in \{0,1\}$, using the relations (R1), (R2), (R3). (<u>Hint</u>: use induction).

<u>Lemma L3-1</u> The set $G = \{ R_\psi T^n \mid \psi \in [0, 2\pi), n \in \{0,1\} \}$ forms a subgroup of $\mathrm{Isom}(S^1, d_a)$.

<u>Proof</u> $id = R_0 T^0$, and G is closed under composition by the exercise. Finally notice that $T^n R_\theta = R_{(-1)^n\theta} T^n$ and so

$$\begin{aligned}(R_\psi T^n)\cdot(R_{(-1)^{n+1}\psi} T^n) &= R_\psi T^n R_{(-1)^{n+1}\psi} T^n \\ &= R_\psi R_{(-1)^n(-1)^{n+1}\psi} T^n T^n \\ &= R_\psi R_{-\psi} (T^2)^n \\ &= R_0 (id)^n = id\end{aligned}$$

That is, $(R_\psi T^n)^{-1} = R_{(-1)^{n+1}\psi} T^n$ is again in G, so G is a subgroup. □

This almost, but not quite, answers Q1. We know everything that looks like F can be written as a product $R_\psi T^n$, but how do we know if are there redundancies, i.e. $R_\psi T^n = R_\theta T^m$ with $(\psi, n) \neq (\theta, m)$?

<u>Lemma L3-2</u> If $\theta, \psi \in [0, 2\pi)$ and $m, n \in \{0,1\}$ then $R_\psi T^n = R_\theta T^m$ if and only if $\theta = \psi$ and $m = n$. In other words, we have a bijection

$$[0, 2\pi) \times \{0,1\} \longrightarrow G, \quad (\theta, n) \mapsto R_\theta T^n$$

<u>Proof</u> Firstly, notice that if two matrices $A, B \in M_2(\mathbb{R})$ induce functions $\mathbb{R}^2 \to \mathbb{R}^2$ by left multiplication which restrict to functions $f_A, f_B : S^1 \to S^1$ which agree $f_A = f_B$, then $A = B$ as matrices, since $f_A(1,0)$ is the first column of A and $f_A(0,1)$ is the second column. So

$$R_\varphi T^n = R_\theta T^m \text{ as functions } S^1 \to S^1$$

$$\iff R_\varphi T^n = R_\theta T^m \text{ as } 2\times 2 \text{ matrices.}$$

But if these matrices are equal their determinants are equal, and

$$\det(R_\varphi T^n) = \det(R_\varphi)\det(T)^n$$
$$= 1 \cdot (-1)^n$$

So if $R_\varphi T^n = R_\theta T^m$ then $(-1)^n = (-1)^m$ and hence $m = n$.
Now, also observe that as matrices

$$R_\varphi T^n = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$$
$$= \begin{pmatrix} \cos\varphi & (-1)^{n+1}\sin\varphi \\ \sin\varphi & (-1)^n\cos\varphi \end{pmatrix}$$

Looking at the first column, we deduce that

$$R_\varphi T^n = R_\theta T^m \implies (\cos\varphi, \sin\varphi) = (\cos\theta, \sin\theta)$$
$$\iff \theta = \varphi \qquad (\text{given both lie in } [0, 2\pi)). \quad \square$$

Notice that this completely answers Q1: given a product F of R_θ's and T's, if you want to determine whether it is equal to a particular R_θ, or T, or more generally if it is equal to some <u>other</u> product G of the same type, you need only follow the algorithm:

1. using the relations (R1), (R2), (R3) write $F = R_\psi T^n$ and $G = R_\theta T^m$ for some $(\psi, n), (\theta, m) \in [0, 2\pi) \times \{0,1\}$.

2. then $F = G \iff R_\psi T^n = R_\theta T^m \iff \psi = \theta$ and $m = n$.

<u>Proposition L3-3</u> $G = \mathrm{Isom}(S^1, d_a)$.

<u>Proof</u> Let $F: S^1 \longrightarrow S^1$ be an isometry, with respect to d_a. Let us write $\langle \underline{v}, \underline{w} \rangle$ for the dot product $\underline{v} \cdot \underline{w}$ in $\mathbb{R}^2$. First of all observe

<u>Claim 1</u>: if $\underline{v}, \underline{w} \in S^1$ then $\langle F\underline{v}, F\underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle$.

<u>Proof of claim</u> if $\underline{v} = \Phi(\theta)$ and $\underline{w} = \Phi(\theta')$ then $\langle \underline{v}, \underline{w} \rangle = \cos(\theta - \theta')$ while by hypothesis if $F\underline{v} = \Phi(\psi)$, $F\underline{w} = \Phi(\psi')$

$$\begin{aligned} \langle F\underline{v}, F\underline{w} \rangle &= \cos(\psi - \psi') \\ &= \cos(d_a(F\underline{v}, F\underline{w})) \\ &= \cos(d_a(\underline{v}, \underline{w})) \\ &= \cos(\theta - \theta'). \ \square \end{aligned}$$

Claim 2 If $\underline{v}, \underline{w} \in S^1$ then $\langle F\underline{v}, \underline{w} \rangle = \langle \underline{v}, F^{-1}\underline{w} \rangle$.

Proof This follows from the first claim, since

$$\langle F\underline{v}, \underline{w} \rangle = \langle F\underline{v}, F(F^{-1}\underline{w}) \rangle$$
$$= \langle \underline{v}, F^{-1}\underline{w} \rangle. \quad \square$$

Claim 3 Set $\underline{a} = F(\underline{e}_1)$, $\underline{b} = F(\underline{e}_2)$ where $\underline{e}_1, \underline{e}_2$ are the standard basis vectors. Then with A the matrix with columns $\underline{a}, \underline{b}$

$$F(\underline{v}) = A\underline{v} \qquad \text{for all } \underline{v} \in S^1.$$

Proof By definition this is true for $\underline{v} = \underline{e}_1$ or $\underline{v} = \underline{e}_2$, and for any other $\underline{v} = \lambda \underline{e}_1 + \mu \underline{e}_2$ and $\underline{w} \in S^1$

$$\begin{aligned} \langle F\underline{v}, \underline{w} \rangle &= \langle \underline{v}, F^{-1}(\underline{w}) \rangle \\ &= \lambda \langle \underline{e}_1, F^{-1}(\underline{w}) \rangle + \mu \langle \underline{e}_2, F^{-1}(\underline{w}) \rangle \\ &= \lambda \langle F\underline{e}_1, \underline{w} \rangle + \mu \langle F\underline{e}_2, \underline{w} \rangle \\ &= \langle \lambda \underline{a} + \mu \underline{b}, \underline{w} \rangle \\ &= \langle A\underline{v}, \underline{w} \rangle. \end{aligned}$$

But since this holds for $\underline{w} \in \{\underline{e}_1, \underline{e}_2\}$ we conclude $F\underline{v} = A\underline{v}$. $\square$

Claim 4 Either $A \in SO(2)$ or $TA \in SO(2)$, where $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof By construction A preserves the inner product, and so $A^T A = I$. Thus either $\det(A) = 1$, in which case $A \in SO(2)$ by Exercise L1-5, or $\det(A) = -1$ in which case $\det(TA) = 1$ and $TA \in SO(2)$. $\square$

So we have either $F = R_\theta$ or $TF = R_\theta$, and in the latter case $F = TR_\theta = R_{-\theta}T$, which completes the proof. $\square$

<u>Exercise L3-4</u> Prove that $R_\theta T : S^1 \longrightarrow S^1$ is reflection of S^1 through the straight line which passes through the origin and $(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}))$.
<u>Hint</u>: use relations.

Thus, you have proven every isometry of S^1 is either a <u>rotation</u> or a <u>reflection</u> through some line

<u>Exercise L3-5</u> This exercise revisits the situation of Lecture 1 (observers in the plane and all that) and especially the $SO(2)$ vs. $O(2)$ distinction, in the light of what we have now understood. We will also use the concept of <u>orientation</u> introduced in the first tutorial. Recall that "having the same orientation" is an equivalence relation $\mathcal{B} \sim \mathcal{C}$ on ordered bases $\mathcal{B}, \mathcal{C}$.

(i) Let $F : V \longrightarrow V$ be an invertible linear operator on a finite-dimensional vector space. Prove that precisely one of the following two possibilities is realised:

$$(\mathrm{I}) \quad \forall \mathcal{B} \left(F(\mathcal{B}) \sim \mathcal{B} \right)$$

($\mathcal{B}$ ranges over all ordered bases)

$$(\mathrm{II}) \quad \forall \mathcal{B} \left(F(\mathcal{B}) \nsim \mathcal{B} \right)$$

where $F(\mathcal{B})$ denotes $(F(\underline{b}_1), \ldots, F(\underline{b}_n))$ if $\mathcal{B} = (\underline{b}_1, \ldots, \underline{b}_n)$.
In the first case we say F is <u>orientation preserving</u> and in the latter case we say F is <u>orientation reversing</u>.

(ii) Prove that F is orientation preserving iff. $\det(F) > 0$, and orientation reversing iff. $\det(F) < 0$.

(iii) Define

$$O(n) := \{ X \in M_n(\mathbb{R}) \mid X \text{ is orthogonal, i.e. } X^T X = I_n \}$$

$$SO(n) := \{ X \in O(n) \mid \det(X) = 1 \}.$$

Prove that $X \in O(n)$ if and only if for all $\underline{v}, \underline{w} \in \mathbb{R}^n$

$$(X\underline{v}) \cdot (X\underline{w}) = \underline{v} \cdot \underline{w}.$$

By part (ii), $SO(n)$ are precisely the matrices in $O(n)$ that give rise to orientation preserving linear transformations $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

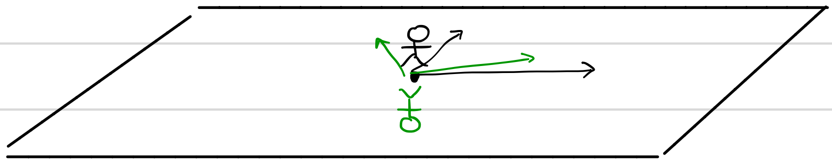
(iv) Prove that $O(n)$ is a group under multiplication, and $SO(n)$ is a subgroup. Produce an element $T \in O(n)$ such that $T^2 = id$ and every element of $O(n)$ not in $SO(n)$ may be written as XT for some $X \in SO(n)$. Thus prove $SO(n) \subseteq O(n)$ is a <u>normal</u> subgroup and that there is a group isomorphism

$$O(n)/SO(n) \cong \mathbb{Z}_2$$

That's the end of the exercise. Some comments linking this to L1 follow overleaf.

<u>Note</u> We have just proven $O(2)$ is the isometry group of S^1. In general, $O(n)$ is the isometry group of S^n, we may return to this later.

Consider two observers O_1, O_2 who are measuring points in the same abstract plane X from <u>different sides</u> (imagine a physical sheet) but at the same point $\underline{O}$, with their axes rotated by some angle relative to one another



<u>Exercise L3-6</u> Prove that there is an element $F \in O(2)$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "m_2"'] \arrow[dr, "m_1"] & \\ \mathbb{R}^2 \arrow[rr, "F"'] & & \mathbb{R}^2 \end{tikzcd}$$

commutes, i.e. a fixed orthogonal matrix which converts O_2's measurements to m_1's measurements.

<u>Upshot</u> We may (and physicists routinely do) identify the set of possible observers with the elements of a symmetry group: if you fix a reference observer O_1 in the plane, for example, there is a bijection

$$\{\text{possible observers}\} \xrightarrow{\cong} O(2)$$

$O_2 \longmapsto$ the group element F which converts O_2's measurements into O_1's measurements.

This explains the fundamental role of groups and group representations in general relativity, quantum mechanics, QFT, ...