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Lecture 6 : Topological spaces

In the previous lecture we explored one reason why we must develop notions of space more abstract than metric spaces: our best theory of (physical) space and time is formulated using a pair of such abstractions, namely <u>quadratic space</u> and <u>topological space</u>. The role of the former is hopefully clear: from the symmetries of a particular quadratic space (Minkowski space) we obtain the transformation relating the measurements of two observers undergoing constant relative linear motion. The purpose of introducing topological spaces is more subtle, and is required to make sense of the idea of the metric tensor as a dynamical quantity in GR.

We will return to this point briefly later, but our chief motivations for introducing topological spaces will be the following:

(1) the concept of <u>continuity</u> is more fundamental than that of <u>distance</u> (even though when we learn about ε, δ the two seem inseparable, they may be separated: liberating continuity from distance is the whole point of topological spaces)

(2) topological spaces are a more convenient "category" (i.e. it is more convenient to build new spaces out of old ones, via quotients, products, gluing ...)

(3) there are natural examples of topological spaces (e.g. Zariski space in algebraic geometry) that do <u>not</u> arise from metric spaces

As we have already mentioned, topological spaces (and the theory of sheaves) make precise the notion of a <u>locally defined quantity</u> in a very powerful way.

<u>Def^n</u> A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a set of subsets of X, such that

(T1) ϕ, X both belong to $\mathcal{T}$,
(T2) if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$,
(T3) if $\{V_i\}_{i \in I}$ is any indexed set with $V_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} V_i \in \mathcal{T}$.

We call such a set $\mathcal{T}$ a <u>topology</u> on X and say that the sets $V \in \mathcal{T}$ are <u>open</u> in the topology. A set $C \subseteq X$ is <u>closed</u> in the topology if there exists $U \in \mathcal{T}$ with $C = X \setminus U$.

<u>Lemma L6-1</u> Let (X, d) be a metric space, and define

$$\mathcal{T}_d = \{ U \subseteq X \mid \forall x \in U \; \exists \varepsilon > 0 \; B_\varepsilon(x) \subseteq U \}$$

where $B_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}$ is the <u>ball</u> of <u>radius</u> ε in X. Then $(X, \mathcal{T}_d)$ is a topological space.

<u>Proof</u> (T1) For $U = \phi$ the predicate is vacuous, so clearly $\phi \in \mathcal{T}_d$.
For $U = X$, take $\varepsilon = 1$ (or anything else) and $B_\varepsilon(x) \subseteq X$.

(T2) If $U, V \in \mathcal{T}_d$ and $x \in U \cap V$ say $B_{\varepsilon_1}(x) \subseteq U$ and $B_{\varepsilon_2}(x) \subseteq V$. Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(x) \subseteq U \cap V$$

since if $d(x,y) < \varepsilon$ then $d(x,y) < \varepsilon_1$ <u>and</u> $d(x,y) < \varepsilon_2$.

(T3) If $V_i \in \mathcal{T}_d$ for all $i \in I$ and $x \in \bigcup_{i\in I} V_i$ then there exists some $i_0 \in I$ with $x \in V_{i_0}$. By hypothesis then there exists $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq V_{i_0}$, and thus $B_\varepsilon(x) \subseteq \bigcup_i V_i$. $\square$

<u>Exercise L6-1</u> If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$ is a subset, then Y is a topological space with the <u>induced topology</u>

$$\mathcal{T}|_Y := \{ U \cap Y \mid U \in \mathcal{T} \}.$$

In the following when we speak of $\mathbb{R}^n$ or intervals $[a,b], (a,b], [a,b), (a,b)$ as topological spaces we will always mean in the first case the topology associated to $(\mathbb{R}^n, d_2)$ and in the second case the subspace topology inherited from $\mathbb{R}$.

<u>Remark</u> (1) Some sets are both open and closed (clopen!) e.g. $\emptyset, X$. But for example $\{0\} \subseteq \mathbb{R}$ is closed but not open, and its complement is therefore open but not closed.

(2) It is not necessarily true that arbitrary intersections of open sets are open, since e.g. in $\mathbb{R}$ we have

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, \tfrac{1}{n}\right) = \{0\}.$$

(3) By induction any <u>finite</u> intersection of open sets is open.

<u>Exercise L6-2</u> Prove that $(S^1, d_a), (S^1, d_2)$ are <u>not</u> isometric, but that $\mathcal{T}_{d_a} = \mathcal{T}_{d_2}$ i.e. in the associated topologies on S^1 the same sets are declared open.

<u>Def^n</u> A topological space $(X, \mathcal{T})$ is <u>metrisable</u> if there exists a metric d on X with $\mathcal{T} = \mathcal{T}_d$.

not all points $\{x\}$ are closed sets!

<u>Exercise L6-3</u> Prove that $X = \{0,1\}$ with $\mathcal{T} = \{\phi, X, \{1\}\}$ is a topological space. This is called the <u>Sierpiński space</u> and is usually denoted Σ. Prove Σ is <u>not</u> metrisable.

So not every topological space is metrisable, and moreover those that are may have their topology induced by more than one metric.

<u>Exercise L6-4</u> Classify all the topologies on the set $X = \{0,1\}$.

<u>Def^n</u> The <u>one-point space</u> is $X = \{*\}$ with its unique topology.

<u>Def^n</u> Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on X we say $\mathcal{T}_1$ is <u>finer</u> than $\mathcal{T}_2$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2$ (i.e. more sets are open in $\mathcal{T}_1$). The <u>discrete topology</u> on X declares <u>every</u> set to be open, while the <u>indiscrete topology</u> is $\{\phi, X\}$ i.e. only ϕ, X are open. Clearly the discrete topology is finer than any topology, and any topology is finer than the indiscrete topology.

<u>Def^n</u> Let $(X,\mathcal{T}), (Y,\mathcal{S})$ be topological spaces. A <u>continuous map</u> $f : (X,\mathcal{T}) \longrightarrow (Y, \mathcal{S})$ is a function $f : X \longrightarrow Y$ with the property that

$$\forall V \subseteq Y \left(V \in \mathcal{S} \Rightarrow f^{-1}(V) \in \mathcal{T} \right)$$

i.e. $\{x \in X \mid f(x) \in V\}$

We denote by $\mathrm{Cts}((X,\mathcal{T}),(Y,\mathcal{S}))$ (or just $\mathrm{Cts}(X,Y)$ if the topologies are clear) the set of all continuous functions $(X,\mathcal{T}) \longrightarrow (Y,\mathcal{S})$.

Exercise L6-5 The identity function is continuous, and the composite of continuous functions is continuous (we say that topological spaces form a category)

Exercise L6-6 There is a bijection $Cts(\{*\}, X) \longrightarrow X$ sending a function $f : \{*\} \longrightarrow X$ to $f(*) \in X$.

Lemma L6-2 Let $(X, \mathcal{T})$ be a topological space. There is a bijection (Σ being the Sierpiński space)

$$\begin{aligned} Cts(X, \Sigma) &\longrightarrow \mathcal{T} \\ f &\longmapsto f^{-1}(\{1\}). \end{aligned} \qquad (*)$$

Proof Since $\{1\}$ is open the function given is well-defined. To define the inverse, let $U \subseteq X$ be open and take its characteristic function

$$\chi_U : X \longrightarrow \{0,1\}$$

$$\chi_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{else.} \end{cases}$$

We claim this is continuous when we give $\{0,1\}$ the Sierpiński topology. We need to prove $\chi_U^{-1}(\phi)$, $\chi_U^{-1}(\{0,1\})$ and $\chi_U^{-1}(\{1\})$ are open. But these sets are ϕ, X, U respectively, so χ_U is continuous, and as $\chi_U^{-1}(\{1\}) = U$ this gives the desired two-sided inverse to $(*)$. $\square$

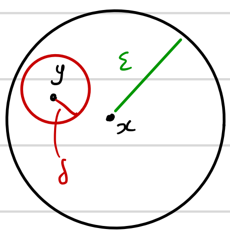
Upshot: the purpose of a topology $\mathcal{T}$ is to tell you which functions out of X are continuous. If you know that, you can recover the topology.

Lemma L6-3 If (X,d) is a metric space and $\mathcal{T}$ the associated topology, then

(i) $B_\varepsilon(x) \in \mathcal{T}$ for all $x \in X$, $\varepsilon > 0$.

(ii) every $U \in \mathcal{T}$ is a union of a set of such open balls.

Proof (i) Given $y \in B_\varepsilon(x)$ we have to find δ s.t. $B_\delta(y) \subseteq B_\varepsilon(x)$.
This is equivalent to finding δ s.t. $d(z,y) < \delta \implies d(z,x) < \varepsilon$.
But we know for any z that



$$d(z,x) \leq d(z,y) + d(y,x)$$

so taking $\delta < \varepsilon - d(x,y)$ will ensure that $B_\delta(y) \subseteq B_\varepsilon(x)$. $\square$

Lemma L6-4 Let (X, d_X) and (Y, d_Y) be metric spaces and $\mathcal{T}_X, \mathcal{T}_Y$ the associated topologies. A function $f : X \longrightarrow Y$ is continuous if and only if

$$\forall x \in X \; \forall \varepsilon > 0 \; \exists \delta > 0 \left(y \in B_\delta(x) \Rightarrow f(y) \in B_\varepsilon(f(x)) \right) \qquad (6.1)$$

i.e. $d_X(x,y) < \delta \Rightarrow d_Y(fx, fy) < \varepsilon$

Proof Suppose (6.1) holds and let $V \subseteq Y$ open be given. To prove $f^{-1}(V)$ is open we have to take a given $x \in f^{-1}(V)$ and produce a $\tau > 0$ with $B_\tau(x) \subseteq f^{-1}(V)$.
But $f(x) \in V$ and V is open, so by def^N there is $\varepsilon > 0$ with $B_\varepsilon(fx) \subseteq V$.
But by (6.1) therefore, we may find $\delta > 0$ with

$$B_\delta(x) \subseteq f^{-1}(B_\varepsilon(fx)) \subseteq f^{-1}V.$$

Taking $\tau = \delta$ we are done.

In the opposite direction, suppose f is continuous, and that $x \in X$, $\varepsilon > 0$ are given. Now by the previous lemma $B_\varepsilon(fx) \in \mathcal{T}_Y$ and hence $f^{-1}B_\varepsilon(fx) \in \mathcal{T}_X$. But this means that there exists $\delta > 0$ s.t.

$$B_\delta(x) \subseteq f^{-1}B_\varepsilon(fx)$$

or what is the same,

$$d(z,x) < \delta \implies d(fz, fx) < \varepsilon. \quad \square$$

Now, you checked in kindergarten that various functions $\mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous in the sense of (6.1), which is the usual continuity of (multi-variable) calculus. We will freely use that these functions are, as a consequence of the lemma, also continuous in the new sense. For example, $\sin(x^2y+t)$ describes a continuous map $\mathbb{R}^3 \longrightarrow \mathbb{R}$.

To deal with continuity of functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ with multiple components, it will be easier to introduce the <u>product space</u> $X \times Y$ which we will do next lecture, along with <u>quotient spaces</u> and <u>gluing</u>, which are all convenient ways of generating new spaces from known ones.

<u>Exercise L6-7</u> Let X be a topological space, and $x \in X$ a point. Let

$$\mathcal{G}_x = \{ (U,f) \mid U \text{ is open}, x \in U \text{ and } f: U \longrightarrow \mathbb{R} \text{ is cts} \}.$$

Prove that $\sim$ defined by $(U,f) \sim (V,g) \iff \exists W \subseteq U \cap V$ (W is open, $x \in W$ and $f|_W = g|_W$) is an equivalence relation on $\mathcal{G}_x$. An equivalence class $[(U,f)] \in \mathcal{G}_x/\sim$ is called a <u>germ</u> of a real-valued function at x.

Lipschitz equivalence Two metrics d_1, d_2 on X are Lipschitz equivalent if there exist $h, k > 0$ such that for any $x, y \in X$

$$h d_2(x,y) \leq d_1(x,y) \leq k d_2(x,y).$$

Exercise L6-8 Check this is an equivalence relation on metrics, $d_1 \sim d_2$.

Exercise L6-9 Prove that if $d_1 \sim d_2$ then the induced topologies are the same, i.e. $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$

Exercise L6-10 Recall the metrics d_1, d_∞ on $\mathbb{R}^2$

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$
$$d_\infty(\underline{x},\underline{y}) = \max\{ |x_i - y_i| \mid 1 \leq i \leq n \}$$

Prove that d_1, d_2, d_∞ are all Lipschitz equivalent, so $(\mathbb{R}^2, \mathcal{T}_{d_1}) = (\mathbb{R}^2, \mathcal{T}_{d_2}) = (\mathbb{R}^2, \mathcal{T}_{d_\infty})$.

(Hint: $d_1(x,y) \geqslant d_2(x,y) \geqslant d_\infty(x,y) \geqslant n^{-1/2} d_2(x,y) \geqslant n^{-1} d_1(x,y)$)

Exercise L6-11 Consider the topological space $(X, \mathcal{T})$ with $X = [0,1]$ and $\mathcal{T} = \{ \emptyset, X, [0, \frac{1}{2}], (\frac{1}{2}, 1] \}$. Classify all the continuous functions $X \longrightarrow \mathbb{R}$.

Exercise L6-12 Given a set X with the discrete metric d (see Ex. L2-1) describe the associated topology $\mathcal{T}_d$.