Lecture 7: Constructing topological spaces

updated 10/8/21

The aim of today's lecture is to learn a set of tools for constructing new topological spaces from old ones. A useful organising principle in this endeavour will be the idea of a <u>universal property</u>, which is a concept from category theory. We will begin with products of spaces, for which it will be convenient to use the concept of a <u>basis</u>.

<u>Def^n</u> Let $(X, \mathcal{J})$ be a topological space. A set $\mathcal{B} \subseteq \mathcal{J}$ of open sets is called a <u>basis</u> for the topology $\mathcal{J}$ if for every pair (x, U) where $U \in \mathcal{J}$ and $x \in U$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$.

<u>Exercise L7-1</u> (i) Prove that $\mathcal{B} \subseteq \mathcal{J}$ is a basis iff. every $U \in \mathcal{J}$ can be written as the union set of a subset $\mathcal{C} \subseteq \mathcal{B}$ (i.e. as a union of sets in $\mathcal{B}$, including infinite unions and the "empty union" which is $\emptyset$).

(ii) If $\mathcal{B}$ is a basis for $\mathcal{J}$ and $f : Y \longrightarrow X$ is a function with Y a topological space, then f is continuous iff. $f^{-1}(B) \in \mathcal{J}_Y$ $\forall B \in \mathcal{B}$.

(iii) If (X,d) is a metric space then $\{ B_\varepsilon(x) \mid x \in X, \varepsilon > 0 \}$ is a basis for the associated topology.

<u>Lemma L7-1</u> Let X be a set and $\mathcal{B}$ a collection of subsets of X satisfying

(B1) For each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

(B2) Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

$B_1 \cap B_2$ is a union of sets from $\mathcal{B}$.
$\quad = \bigcup_{x \in B_1 \cap B_2} B_3^x$

Then there is a <u>unique</u> topology $\mathcal{J}$ on X for which $\mathcal{B}$ is a basis. We call $\mathcal{J}$ the topology <u>generated by $\mathcal{B}$</u>.

<u>Proof</u> It is clear that $\mathcal{B}$ can be a basis for <u>at most</u> one topology, so we need only prove that the natural candidate

$\{x \mid \exists C \in \mathcal{C} \text{ with } x \in C\}$

$$\mathcal{T} = \{ V \subseteq X \mid \exists \mathcal{C} \subseteq \mathcal{B} (V = \bigcup \mathcal{C}) \}$$

is in fact a topology. We verify each axiom in turn:

(T1) $\mathcal{C} = \phi$ gives $\phi \in \mathcal{T}$ and $\mathcal{C} = \mathcal{B}$ gives $X = \bigcup \mathcal{B}$ by (B1) so $X \in \mathcal{T}$.

(T2) Suppose $V_1, V_2 \in \mathcal{T}$ with $V_1 = \bigcup \mathcal{C}_1, V_2 = \bigcup \mathcal{C}_2$. We claim $V_1 \cap V_2 \in \mathcal{T}$. Let $\mathcal{C} \subseteq \mathcal{B}$ be the set of all $B \in \mathcal{B}$ with $B \subseteq V_1 \cap V_2$. If $x \in V_1 \cap V_2$ then there is $B_1 \in \mathcal{C}_1$, $B_2 \in \mathcal{C}_2$ with $x \in B_1 \subseteq V_1$, $x \in B_2 \subseteq V_2$ and by (B2), there is $B_3 \subseteq B_1 \cap B_2$ with $x \in B_3$. But then $B_3 \subseteq B_1 \cap B_2 \subseteq V_1 \cap V_2$, so $B_3 \in \mathcal{C}$. This proves $V_1 \cap V_2 = \bigcup \mathcal{C}$, so $V_1 \cap V_2 \in \mathcal{T}$.

(T3). If $\{V_i\}_{i \in I}$ are open, say $V_i = \bigcup \mathcal{C}_i$, then with $\mathcal{C} = \bigcup_i \mathcal{C}_i$ we have $\bigcup_i V_i = \bigcup \mathcal{C}$, so we are done. $\square$

<u>Remark</u> Let $\{X_i\}_{i \in I}$ be an indexed family of sets. Recall the <u>product</u> is

$$\prod_{i \in I} X_i = \{ (x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i \in I \}.$$

For example $\prod_{n \in \mathbb{N}} \mathbb{R}$ is just the set of all real sequences $(a_0, a_1, a_2, \ldots)$. If we take $X_i = Y$, some fixed set, for all $i \in I$ then $\prod_{i \in I} Y$ is just the set of functions $f : I \longrightarrow Y$, just as a sequence of real numbers can be viewed as a function $\mathbb{N} \longrightarrow \mathbb{R}$.

<u>Def^n</u> Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces. The <u>product space</u> $\prod_{i\in I} X_i$ is the usual product set with the topology generated by the basis $\mathcal{B}$ consisting of sets

$$\prod_{i\in I} U_i = \{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \in U_i \text{ for all } i \}$$

where each $U_i \subseteq X_i$ is open and the set $\{ i\in I \mid U_i \neq X_i \}$ is finite. More precisely the basis is

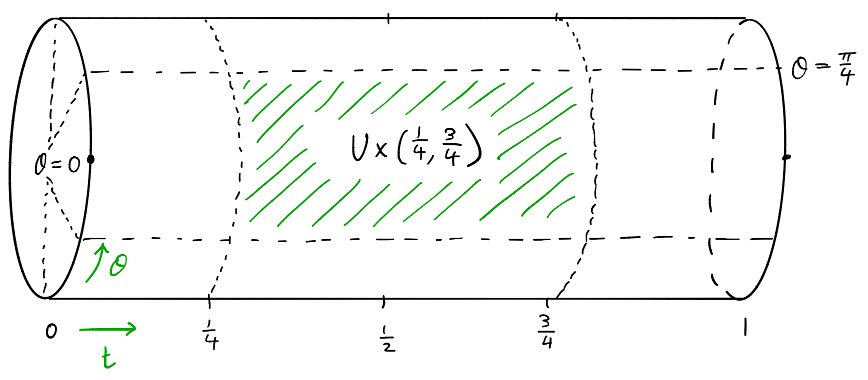
$$\mathcal{B} = \left\{ \prod_{i\in I} U_i \;\middle|\; \begin{array}{l} U_i \subseteq X_i \text{ is open for all } i\in I \\ \text{and } \{ i\in I \mid U_i \neq X_i\} \text{ is finite} \end{array} \right\}$$

<u>Exercise L7-2</u> Prove that the $\prod_i U_i$ as defined above satisfy (B1), (B2), so that the topology on $\prod_{i\in I} X_i$ is well-defined.

<u>Remark</u> As usual we write $X\times Y$ for $\prod_{i\in I} X_i$ where $I=\{1,2\}$, $X_1 = X$, $X_2 = Y$. However the general product notation is useful because it avoids questions like: is $X\times Y\times Z = X\times(Y\times Z)$ or $(X\times Y)\times Z$? which would arise if we insisted only on defining binary products. We say:

$$X\times Y\times Z := \prod_{i\in I} X_i \qquad I=\{1,2,3\}\ X_1 = X,\ X_2=Y,\ X_3 = Z.$$

<u>Remark</u> In the case where I is finite $\{ i\in I \mid U_i \neq X_i \}$ is always finite, so the basis consists of all products of open sets $\prod_{i\in I} U_i$.
For example if S^1 denotes the topological space associated to (S^1, d_a) then $U = B_{\frac{\pi}{4}}((1,0))$ is open in S^1 and so $U\times(\frac{1}{4},\frac{3}{4}) \subseteq S^1 \times [0,1]$ is open (see overleaf).



$S^1 \times [0,1]$
(θ, t)

<u>Def^n</u> A continuous map $f : X \longrightarrow Y$ is a <u>homeomorphism</u> (or isomorphism) if there is a continuous map $g : Y \longrightarrow X$ with $f \circ g = id_Y$, $g \circ f = id_X$.

<u>Exercise L7-3</u> Prove a continuous map $f : X \longrightarrow Y$ is a homeomorphism iff. it is a bijection and $f(U) \subseteq Y$ is open whenever $U \subseteq X$ is open. Give a counterexample to show not all continuous bijections are homeomorphisms.

<u>Exercise L7-3½</u> Let X be a set. Prove (i) that if $E_i \subseteq X \times X$ is an equivalence relation then so is $\bigcap_{i \in I} E_i$.

(ii) Given a subset $Q \subseteq X \times X$ prove that

$$E = \bigcap \{ Y \subseteq X \times X \mid Y \text{ is an equiv. rel}^N \text{ \& } Y \supseteq Q \}$$

is an equivalence relation. This is the equivalence relation <u>generated by Q</u>.

(iii) Given Q, E as above, suppose $f : X \longrightarrow Y$ is a function with $f(x_1) = f(x_2)$ whenever $(x_1, x_2) \in Q$. Prove $f(x_1) = f(x_2)$ for all $(x_1, x_2) \in E$.

<u>Exercise L7-4</u> (i) Prove $\mathbb{R}^n$ (with the metric topology) is <u>equal</u> as a topological space to the product of n copies of $\mathbb{R}$, in the above sense.

(ii) Is the space $\mathbb{R}^\omega := \prod_{n \in \mathbb{N}} \mathbb{R}$ metrisable? Prove it, either way.

<u>Exercise L7-5</u> Prove that $\pi_j : \prod_i X_i \longrightarrow X_j$ defined by $\pi_j\left((x_i)_{i \in I}\right) = x_j$ is continuous. We call π_j the <u>j-projection</u>.

<u>Lemma L7-2</u> (Universal property of the product) Let $\{X_i\}_{i\in I}$ be a family of topological spaces and Y another topological space. There is a bijection

$$\mathrm{Cts}(Y, \textstyle\prod_{i\in I} X_i) \xrightarrow{\Phi} \prod_{i\in I} \mathrm{Cts}(Y, X_i)$$
$$\Phi(f) = (\pi_i \circ f)_{i\in I}.$$

That is, given $f_i : Y \longrightarrow X_i$ continuous there is a <u>unique</u> continuous map $f : Y \longrightarrow \prod_i X_i$ with $\pi_i \circ f = f_i$ for all $i \in I$.

<u>Proof</u> Since the π_i are continuous and composites of continuous functions are continuous, Φ is at least well-defined. We define a candidate inverse Φ^{-1} by sending a family of continuous functions $(f_i)_i$ to the function f defined by $f(y) = (f_i(y))_{i\in I} \in \prod_{i\in I} X_i$.

If we can show f is continuous we are done, as clearly Φ, Φ^{-1} are mutually inverse. By Ex L7-1 (ii) it suffices to show f^{-1} sends open sets <u>in the basis</u> to open sets in Y. Suppose $U_i \subseteq X_i$ is open and set $Q = \{i \in I \mid U_i \neq X_i\}$ which is finite. Then

$$\begin{aligned} f^{-1}(\textstyle\prod_{i\in I} U_i) &= \{y \in Y \mid f(y) \in \textstyle\prod_i U_i\} \\ &= \{y \in Y \mid f_i(y) \in U_i \text{ for all } i \in I\} \\ &= \{y \in Y \mid f_i(y) \in U_i \text{ for } i \in Q\} \\ &= \textstyle\bigcap_{i\in Q} f_i^{-1}(U_i) \end{aligned}$$

which as a finite intersection of open sets, is open. □

<u>Exercise L7-5½</u> Given topological spaces $\{X_i\}_{i\in I}$ let $\prod^{alt}_{i\in I} X_i$ denote the set $\prod_{i\in I} X_i$ with the <u>alternate topology</u> which has as a basis the sets $\prod_{i\in I} U_i$ where $U_i \subseteq X_i$ is open for all $i \in I$ (i.e. we do not impose the condition that $\{i \in I \mid U_i \neq X_i\}$ is finite). Prove that this is a valid basis, but give a counterexample to show $\prod^{alt}_{i\in I} X_i$ does <u>not</u> have the universal property of Lemma L7-2 (obviously I must be infinite).

<u>Def^n</u> Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces. The <u>disjoint union</u> or <u>coproduct space</u> $\coprod_{i\in I} X_i$ is the disjoint union set

$$\coprod_{i\in I} X_i = \bigcup_{i\in I} \{i\} \times X_i$$

with the topology given by the subsets

$$\coprod_{i\in I} U_i = \{ (i,x) \mid i \in I, x \in U_i \} \qquad (5.1)$$

where $U_i \subseteq X_i$ is open. More precisely the topology $\mathcal{T}$ is

$$\mathcal{T} = \Big\{ \coprod_{i\in I} U_i \;\Big|\; U_i \subseteq X_i \text{ open for all } i \in I \Big\}.$$

<u>Exercise L7-6</u> (i) Prove (5.1) is a topology and prove $i_j : X_j \longrightarrow \coprod_{i\in I} X_i$ sending x to (j,x) is continuous.

(ii) (Universal property) Prove that for any space Y there is a bijection

$$\operatorname{Cts}\Big(\coprod_{i\in I} X_i, Y\Big) \xrightarrow{\cong} \prod_{i\in I} \operatorname{Cts}(X_i, Y).$$

<u>Note</u> taking $Y = \Sigma$ (Sierpiński) we see that the universal property dictates the topology.

<u>Def^n</u> Let X be a topological space and $\sim$ an equivalence relation on X. The <u>quotient space</u> $X/\sim$ is the set of equivalence classes with the topology given by ($\rho : X \to X/\sim$ denotes the quotient map)

$$\mathcal{T} = \{ U \subseteq X/\sim \mid \rho^{-1}(U) \text{ is open in } X \}.$$

Clearly then ρ is a continuous map.

<u>Exercise L7-7</u> Prove this is a topology on $X/\sim$ and that for any space Y and any continuous $f : X \to Y$ s.t. $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$, there is a <u>unique</u> continuous map F making the diagram below commute:

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "f"'] & X/\sim \arrow[d, dashed, "F"] \\ & Y \end{tikzcd}$$

<u>Def^n</u> Suppose given a pair of continuous maps $f : X \to Y$, $g : X \to Z$. The <u>pushout</u> of the (unordered) pair f, g is the space

$$Y \amalg_X Z := (Y \amalg Z)/\sim$$

where $\sim$ is the smallest equivalence relation containing the pairs $(f(x), g(x))$ for all $x \in X$.

<u>Exercise L7-8</u> Prove that the maps $\iota_Y : Y \to Y \amalg_X Z$, $\iota_Z : Z \to Y \amalg_X Z$ defined resp. by $\iota_Y(y) = [y]$ and $\iota_Z(z) = [z]$ are continuous.

<u>Lemma L7-3</u> (Universal property of the pushout) Let f, g as above be given. Given continuous maps $u : Y \longrightarrow W$ and $v : Z \longrightarrow W$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "u"] \\
Z \arrow[r, "v"'] & W
\end{tikzcd}$$

commutes (i.e. $u \circ f = v \circ g$) there is a unique continuous map $t : Y \amalg_X Z \longrightarrow W$ such that the two marked triangles in the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\iota_Y"'] \arrow[ddr, bend left, "u"] \arrow[ddr, phantom, "\circledast"] & \\
Z \arrow[r, "\iota_Z"] \arrow[drr, bend right, "v"'] \arrow[drr, phantom, "\circledast"] & Y \amalg_X Z \arrow[dr, dashed, "t"] & \\
 & & W
\end{tikzcd}$$

<u>Proof</u> First let us prove uniqueness. If $t \circ \iota_Z = t' \circ \iota_Z$ and $t \circ \iota_Y = t' \circ \iota_Y$ then we calculate

$$t([y]) = t(\iota_Y(y)) = t'(\iota_Y(y)) = t'([y])$$
$$t([z]) = t(\iota_Z(z)) = t'(\iota_Z(z)) = t'([z])$$

and since every element of $Y \amalg_X Z$ is either $[y]$ for some $y \in Y$ or $[z]$ for some $z \in Z$, this proves $t = t'$. So it suffices to prove existence.

By Exercise L7-7 to define $t : Y \amalg_X Z \to W$ continuous it is enough to define $t' : Y \amalg Z \to W$ continuous s.t. $t'(f(x)) = t'(g(x))$ for all x. But by Exercise L7-6 the functions u, v determine a continuous $Y \amalg Z \to W$ with this property, since by hypothesis $u \circ f = v \circ g$. This shows

$$\begin{aligned} t : Y \amalg_X Z &\longrightarrow W \\ t([y]) &= u(y) \\ t([z]) &= v(z) \end{aligned}$$

is a well-defined continuous map, and clearly $t \circ \iota_Y = u$, $t \circ \iota_Z = v$, so we are done. $\square$

<u>Example L7-1</u> (The circle) Let $\sim$ be the equivalence relation on $[0,1]$ generated by $0 \sim 1$. There is a bijection

$$\mathrm{Cts}([0,1]/\sim, X) \cong \{ f : [0,1] \to X \text{ cts} \mid f(0) = f(1) \}.$$

Give a homeomorphism $S^1 \cong [0,1]/\sim$.

<u>Exercise L7-10</u> Let us write $\{*_0, *_1\} := \{*\} \amalg \{*\}$ and $f : \{*_0, *_1\} \to [0,1]$ for the inclusion of the endpoints $f(*_0) = 0$, $f(*_1) = 1$. We may form the pushout

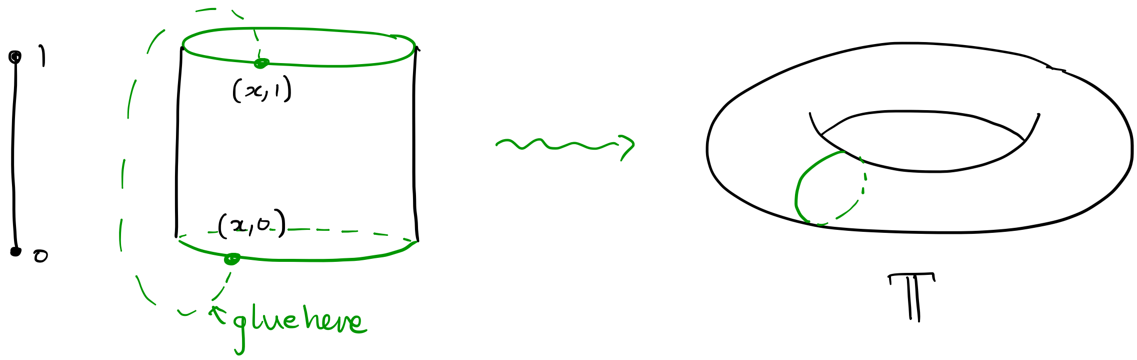
$$\begin{array}{ccc} \{*\} \amalg \{*\} & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ [0,1] & \longrightarrow & P := [0,1] \amalg_{\{*_0, *_1\}} \{*\} \end{array}$$

Prove that $P \cong S^1$.

<u>Example L7-2</u> (The torus) Let $C := S^1 \times [0,1]$ be the cylinder, and let $f : S^1 \to C$ be the "bottom" and $g : S^1 \to C$ the "top", i.e.

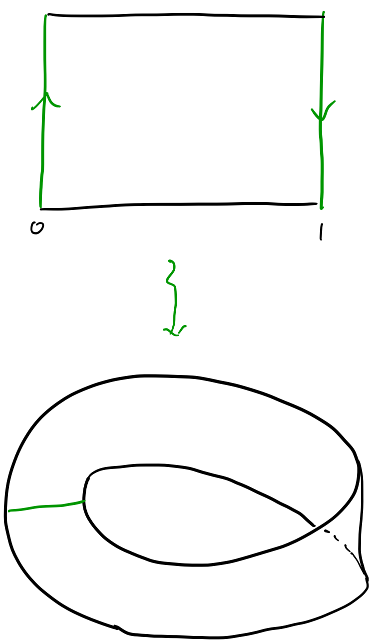
$$f(x) = (x,0) \qquad \forall x \in S^1$$
$$g(x) = (x,1) \qquad \forall x \in S^1$$

We define $\mathbb{T} := C/\sim$ where $\sim$ is the equivalence relation generated by the pairs $f(x) \sim g(x)$ for all $x \in S^1$. This is the <u>torus</u>.



<u>Example L7-3</u> (Möbius strip) Let M be $[0,1] \times [0,1]/\sim$ where $\sim$ is the equivalence relation generated by

$$(1,\lambda) \sim (0, 1-\lambda) \qquad 0 \leq \lambda \leq 1.$$



This is called the Möbius strip.

We have now developed a good set of tools for constructing new spaces and reasoning about them. But we lack any understanding of how to compare the resulting objects. For example

Question How can we prove $M \not\cong S^1 \times [0,1]$ (where $\cong$ means homeomorphic)

or $\mathbb{T} \not\cong S^2$

$\mathbb{R} \not\cong S^1$

or for that matter $\mathbb{R}^2 \not\cong \mathbb{R}^3$!

Quick answers

- $S^1 \times [0,1]$ is orientable but M is not
  (but we have to show "orientability" is purely topological)
- $\mathbb{T}$ contains "nontrivial" loops, S^2 does not
  (this leads us to think about the fundamental group)
- S^1 is compact but $\mathbb{R}$ is not
- $\mathbb{R}^n \not\cong \mathbb{R}^m$ when $n \neq m$ but this is not trivial, e.g.

Peano: there is a surjective continuous map $[0,1] \to [0,1]^2$
(I told you not to trust your intuition about top. spaces!)

The problem of distinguishing topological spaces is a deep one, and we will only discuss the most elementary aspects in this course. Analogous problems for spaces with additional structure are central to e.g. algebraic geometry, differential geometry, algebraic topology, ... see the final page of these notes for a brief intro.

<u>Exercise L7-11</u> In the above fashion, explain how to glue two copies of the disk $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ along their boundary circles to obtain a space homeomorphic to $S^2 \subseteq \mathbb{R}^3$

<u>Exercise L7-12</u> (i) Prove (a,b) is homeomorphic to $\mathbb{R}$ for any $a<b$.

(ii) Prove $\mathbb{T} \cong S^1 \times S^1$. ($\mathbb{T}$ defined as in Example L7-2)

<u>Exercise L7-13</u> Characterise the open subsets of the torus containing a point $[(x,1)]$ on the "glued edge". (<u>Hint</u>: use the Sierpinski space).

<u>Exercise L7-14</u> Prove that any linear transformation $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ is continuous (you can try using the metric, but is much easier using products and Ex L7-4).

<u>Exercise L7-15</u> Let V be a finite-dimensional vector space. Given a basis $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ we may use the associated isomorphism $V \cong \mathbb{R}^n$ to put a topology $\mathcal{T}_{\mathcal{B}}$ on V. Prove this topology is <u>independent of $\mathcal{B}$</u>. Thus any finite-dimensional real-vector space has a canonical topology. Prove that if V, W are f.d. vector spaces with the canonical topologies, any linear map $V \longrightarrow W$ is continuous.

<u>Exercise L7-16</u> Let X, Y be topological spaces and $A \subseteq X$ a subspace. Prove that a function $f: Y \longrightarrow A$ is continuous iff. $i \circ f: Y \longrightarrow X$ is continuous where $i: A \rightarrow X$ is the inclusion.

<u>Exercise L7-17</u> Given spaces $\{X_i\}_{i \in I}$ and Y, prove a function $Y \xrightarrow{f} \prod_i X_i$ with components $f_i: Y \longrightarrow X_i$ is continuous iff. f_i is continuous for all i.

Def^n For $n \geq 0$ the <u>n-sphere</u> and <u>n-disk</u> are the subspaces

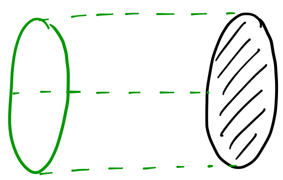
$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \} \subseteq \mathbb{R}^{n+1}$$

$$D^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \} \subseteq \mathbb{R}^n$$

with $S^0 = \{-1, 1\}$ and $D^1 = [-1,1] \subseteq \mathbb{R}$ while $D^0 = \{*\}$.

We denote by $\iota : S^{n-1} \longrightarrow D^n$ the inclusion for $n \geq 1$, e.g.

$$S^0 \xhookrightarrow{\iota} D^1 \qquad\qquad S^1 \xhookrightarrow{\iota} D^2$$



Example L7-4 (Graphs as spaces) Let G be a finite oriented graph, with vertex set V (assumed nonempty) and edge set E. We define a topological space $X(G)$ as follows: let X_0 be V with the discrete topology and let $E = \{e_1, \ldots, e_n\}$. Given an edge $e = (v_1, v_2)$ let $f_e : S^0 \longrightarrow X_0$ be $-1 \mapsto v_1$, $1 \mapsto v_2$ (in fact the space we construct is independent of the orientation). Let $X(G)$ be the pushout

$$\begin{tikzcd}
\coprod_{e \in E} S^0 \arrow[r, "f"] \arrow[d, "\amalg_e \iota"'] & X_0 \arrow[d] \\
\coprod_{e \in E} D^1 \arrow[r] & X(G) := X_0 \amalg_{\amalg_e S^0} \amalg_e D^1
\end{tikzcd}$$

where f restricted to the copy of S^0 indexed by e is f_e.

The space $X(G)$ is a finite set of intervals glued according to G.

Def^n We call a commutative diagram of continuous maps

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"'] \arrow[ur, phantom, "\ulcorner" very near end] & W \end{tikzcd}$$

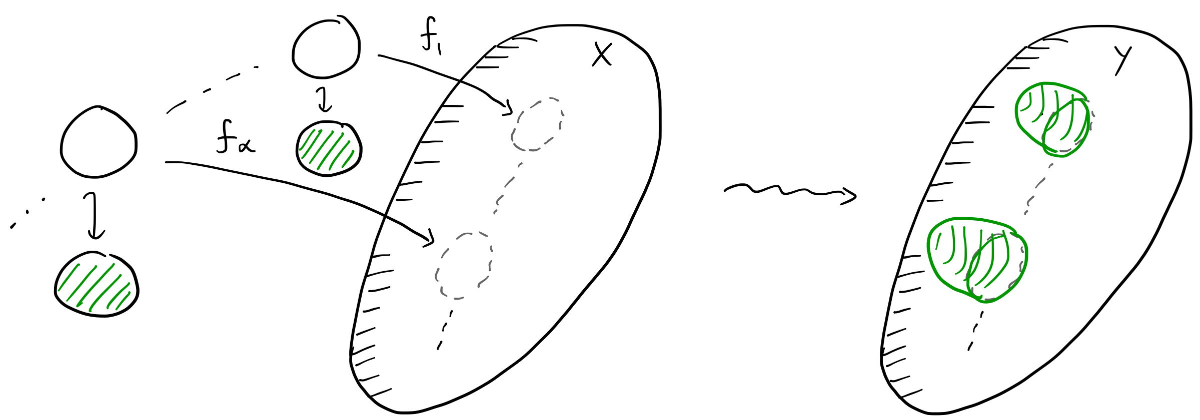
a <u>pushout square</u> (sometimes indicated with $\ulcorner$) if the unique map $t : Y \amalg_X Z \longrightarrow W$ of Lemma L7-3 is a homeomorphism.

Def^n We say a topological space Y is obtained from X by <u>attaching n-cells</u> (for $n \geq 1$) if there is a family of continuous maps $\{ f_\alpha : S^{n-1} \longrightarrow X \}_{\alpha \in \Lambda}$ and a pushout square of the form

$$\begin{tikzcd} \coprod_{\alpha \in \Lambda} S^{n-1} \arrow[r, "f"] \arrow[d, "\amalg_\alpha \iota"'] & X \arrow[d] \\ \coprod_{\alpha \in \Lambda} D^n \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & Y \end{tikzcd}$$

as before, this restricts to f_α on the α-labelled copy of S^{n-1}.

That is Y is obtained from X by <u>gluing in</u> the n-cells D^n along the <u>attaching maps</u> f_α. The set Λ may be empty.



<u>Def^n</u> A topological space X is a <u>finite CW-complex</u> if there is a sequence $X_0, \dots, X_{n-1}, X_n = X$ of topological spaces where X_0 is a finite set with the discrete topology and for $1 \leq i \leq n$ the space X_i is obtained from X_{i-1} by attaching a finite number of i-cells (i.e. the Λ above is finite).
A <u>presentation</u> of X is such a sequence together with the attaching maps $\{f_\alpha : S^{i-1} \longrightarrow X_{i-1}\}_{\alpha \in \Lambda_i}$ used at each stage, i.e. Λ_i indexes i-cells attached.

<u>Example L7-5</u> The space $X(G)$ associated to a graph G is a finite CW-complex with $|V|$ 0-cells (i.e. $|X_0| = |V|$) and $|E|$ 1-cells (i.e. $|\Lambda_1| = |E|$).

<u>Exercise L7-18</u> Present the torus as a finite CW-complex with one 0-cell (i.e. $|\Lambda_0| = 1$), two 1-cells (i.e. $|\Lambda_1| = 2$) and one 2-cell (i.e. $|\Lambda_2| = 1$).

<u>Exercise L7-19</u> Write D^n/S^{n-1} for the quotient space $D^n/\sim$ where $\sim$ is the smallest equivalence relation with $x \sim y$ for all $x, y \in S^{n-1} \subseteq D^n$.
(i) Prove $D^2/S^1 \cong S^2$
(ii)* Prove $D^n/S^{n-1} \cong S^n$ for $n > 2$.
(iii) Prove S^n is a finite CW-complex by attaching a single n-cell to a single 0-cell (i.e. all intermediate stages have Λ empty).

<u>NOTE</u> For this exercise <u>only</u>, you may use that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism (see lectures).

<u>Def^n</u> For $n \geq 1$ define <u>real projective space</u> as the quotient space
$$\mathbb{RP}^n := \left(\mathbb{R}^{n+1} \setminus \{\underline{0}\}\right)/\sim \quad \text{where } (a_0, \dots, a_n) \sim (b_0, \dots, b_n)$$
if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ with $\lambda a_i = b_i$ for all $0 \leq i \leq n$.

<u>Exercise L7-20*</u> Prove $\mathbb{RP}^n$ is a finite CW-complex (there are multiple ways to do this).

Finite CW-complexes will be an excellent source of <u>compact Hausdorff spaces</u> later in the course. They also play a central role in algebraic topology. Further, any compact smooth manifold is homotopy equivalent (a weaker form of equivalence than homeomorphism, but strong enough so invariants like (co) homology agree) to a CW-complex having one k-cell for each critical point of index k of a fixed Morse function (we discussed the <u>index</u> of a critical point in Tutorial #2). This fact is, needless to say, outside the scope of this course, but it is one of many deep and beautiful connections between <u>topology</u> and <u>analysis</u> (where the theory of Morse functions belongs).

<u>Example L7-6</u> Consider the indexed family of spaces $\{X_n\}_{n\geq 0}$ where $X_n = [0,\infty)$ for $n \geq 0$, and let

$$f_n : [0,\infty) \longrightarrow X_n \qquad f(x) = nx.$$

This family determines a function $F : [0,\infty) \longrightarrow \prod_{n\geq 0} X_n$

$$F(x) = (f_n(x))_{n\geq 0} = (nx)_{n\geq 0}$$

which is continuous with respect to the product topology but <u>not</u> the "naive" product topology in which we declare arbitrary products $\prod_n U_n \subseteq \prod_n X_n$ to be open if $U_n \subseteq X_n$ is open (i.e. we drop the finiteness condition). To see this, observe that

$$F^{-1}\big(\textstyle\prod_{n\geq 0}[0,1)\big) = \{x \geq 0 \mid f_n(x) < 1 \text{ for all } n\}$$
$$= \{x \geq 0 \mid x < \tfrac{1}{n} \text{ for all } n\} = \{0\}$$

is <u>not</u> open in $[0,1)$.

<u>Exercise L7-21</u> Given $P > 0$ let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by $x \sim x + P$ for all $x \in \mathbb{R}$. Prove that $\mathbb{R}/\sim \; \cong S^1$ and hence that there is a bijection for any space Y

$$\mathrm{Cts}(S^1, Y) \cong \{ f : \mathbb{R} \longrightarrow Y \mid f \text{ is continuous, and } f(x) = f(x+P) \text{ for all } x \}$$

The circle is the space that <u>represents</u> <u>periodic continuous functions</u>.

Coda: why topological spaces?

Topological spaces are, admittedly, quite abstract. So why do we need them at all? Why not stick with metric spaces? One reason is that not all interesting topological spaces are metrisable. But let us even grant that all the spaces $(X, \mathcal{T})$ we care about are metrisable, i.e. $\mathcal{T} = \mathcal{T}_d$ for some d. <u>Still</u> there are reasons to work with $(X, \mathcal{T})$ rather than (X, d). Here are some:

1. There may be many metrics inducing the same topology, e.g. $(\mathbb{R}^2, \mathcal{T}_{d_2}) = (\mathbb{R}^2, \mathcal{T}_{d_1}) = (\mathbb{R}^2, \mathcal{T}_{d_\infty})$. Working with the topological space is like working with metric spaces modulo this "topological equivalence"

2. Constructions like quotients are awkward for metric spaces, but it is very convenient to build spaces this way.

   Example: compactness results.

3. Many important theorems are more "naturally" proven using topology.

4. Einstein says you should get over global rulers, already.

Aside on invariants of spaces

This lecture we learned the basic constuctions in the theory of topological spaces (products, disjoint unions, quotients and pushouts) and raised the fundamental question of how to tell if two topological spaces X, Y are "the same" i.e. homeomorphic. There are several areas of mathematics that have arisen out of the quest to answer such questions, including homotopy theory and algebraic topology. These subjects are organised around invariants which are quantities (typically numbers, groups or finite dimensional vector spaces) associated to spaces $X \longmapsto I(X)$ such that $X \cong Y \Longrightarrow I(X) = I(Y)$. In particular

$$I(X) \neq I(Y) \Longrightarrow X \not\cong Y.$$

So one accumulates invariants $\{I_1, I_2, \ldots\}$ and given a pair X, Y tries them all until either $I_n(X) \neq I_n(Y)$ (they are not homeomorphic) or one runs out of invariants (so then one might try proving $X \cong Y$ somehow).

Example The fundamental group $\pi_1(X, x)$ and singular homology $H_n(X; \mathbb{Z})$.

The whole point of this game is that X is complicated and $I(X)$ is simple (e.g., it is a number) so comparing $I(X)$ to $I(Y)$ is feasible even if comparing X, Y directly is not. So by definition most of the information in X has been "thrown away" (in principle) in forming $I(X)$ (and this explains why we need many different invariants).

Further reading: Hatcher's "Algebraic topology".