Lecture 8 : Compact spaces I

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In a series of lectures we now develop the notion of <u>compactness</u>, first for metric spaces and then more generally for topological spaces. This is an important <u>finiteness</u> condition for spaces, where the closed interval $[a,b]$ counts as "finite" but $\mathbb{R}$ does not. Recall from calculus that the closed interval has various special properties with respect to continuous functions defined on it, for example:

<u>Extreme Value Theorem</u>: if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous then f is bounded and attains its minimum and maximum, i.e. there exist $c, d \in [a,b]$ such that

$$f(c) \geq f(x) \geq f(d) \qquad \forall x \in [a,b].$$

<u>Uniform Continuity</u> if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous then it is uniformly continuous, i.e. from

$$f \text{ continuous} : \forall x \, \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall y \, \big(|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon \big)$$

we may deduce

$$f \text{ uniformly cts} : \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \, \big(|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon \big).$$

The property of <u>compactness</u> is "responsible" for these and other good properties of the interval, in the sense that these results generalise to any compact subset of a metric space (and in a suitable form, to any compact subspace of a topological space). We study compactness for metric spaces first, but the deep theorems will use only the topology.

Recall the following definitions from real analysis:

<u>Def^n</u> A subset $X \subseteq \mathbb{R}$ is <u>bounded</u> if $X \subseteq [-M, M]$ for some $M > 0$.

<u>Def^n</u> Let $X \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. We call x an <u>adherent point</u> of X if either of the following two equivalent conditions are met:

(i) there is a sequence $(a_n)_{n=0}^{\infty}$, with $a_n \in X$ for all n, converging to x.
(ii) $\forall \varepsilon > 0 \ \exists y \in X (\ |x-y| < \varepsilon \ )$.

The set X is <u>closed</u> if it contains all its adherent points. The same is true in any metric space, see Exercise L8-7.

<u>Lemma L8-1</u> X is closed in this sense iff. it is closed in the metric topology on $\mathbb{R}$.

<u>Proof</u> Suppose X is closed in the metric topology, and that $x \in \mathbb{R}$ is an adherent point of X. We have to show $x \in X$. Suppose not. Since $\mathbb{R} \setminus X$ is open, there is a ball $x \in B_\varepsilon(x) \subseteq \mathbb{R} \setminus X$. But by (ii) above, there exists $y \in X$ with $|x-y| < \varepsilon$ and thus $y \in B_\varepsilon(x)$. But this is a contradiction.

If X is closed in the above sense and $x \notin X$ then x is <u>not</u> an adherent point, so $\exists \varepsilon > 0 \ \forall y \in X (\ |x-y| \geqslant \varepsilon)$, which says $\exists \varepsilon > 0 \ B_\varepsilon(x) \subseteq \mathbb{R} \setminus X$, which shows $\mathbb{R} \setminus X$ is open. $\square$

<u>Example L8-1</u> $[a,b] \subseteq \mathbb{R}$ is closed and bounded.

I assume you have all seen the following proof, but it is worth recalling:

<u>Theorem L8-2</u> (Bolzano–Weierstrass) A subset $K \subseteq \mathbb{R}$ is closed and bounded if and only if every sequence $(a_n)_{n=0}^{\infty}$ in K contains a subsequence which converges to a point in K.

<u>Proof</u> Suppose K is closed and bounded, say $K \subseteq I = [-M, M]$. One of

$$K \cap [-M, 0], \quad K \cap [0, M]$$

must contain a_n for infinitely many n (maybe both do). Let I_1 denote a half of I s.t. $K \cap I_1$ has this property. Bisect I_1 and choose a half I_2 s.t. $K \cap I_2$ contains a_n for infinitely many n. In this way we construct intervals $\{I_j\}_{j \geq 1}$ s.t.

- $I_{j+1} \subseteq I_j \qquad \forall j \geq 1$
- length of $I_j = 2^{-j+1} M \qquad$ (length of I_1 is M)

Choose n_1 with $a_{n_1} \in I_1 \cap K$. Since $\{n \mid a_n \in I_2 \cap K\}$ is infinite we may choose $n_2 > n_1$ with $a_{n_2} \in I_2 \cap K$, and continuing we produce a sequence $(a_{n_j})_{j=1}^{\infty}$ with $a_{n_j} \in I_j \cap K$ for $j \geq 1$. If $i, j \geq k$ then $a_{n_i}, a_{n_j} \in I_k$ and so

$$|a_{n_i} - a_{n_j}| \leq \text{length of } I_k = \frac{1}{2^{k+1}} M$$

Hence $(a_{n_j})_{j=1}^{\infty}$ is Cauchy and converges to $a \in \mathbb{R}$. But a is then an adherent point of K, and since K is closed, we must have $a \in K$.

For the converse, suppose every sequence in K has a subsequence converging to a point in K. We show

- <u>K is bounded</u> otherwise for each $n \geq 1$ let $x_n \in K \setminus [-n, n]$. This must have a subsequence x_{n_i} which converges, say to $x \in K$. Let m be an integer s.t. $x \in (-m, m)$. Let $\varepsilon > 0$ be small enough so $B_\varepsilon(x) \subseteq (-m, m)$. Since $x_{n_i} \to x$ there exists $N > 0$ such that for all $i \geq N$, $x_{n_i} \in B_\varepsilon(x) \subseteq (-m, m)$. For i sufficiently large $n_i > m$ so $x_{n_i} \notin [-n_i, n_i]$. Since $(-m, m) \subseteq [-n_i, n_i]$ this is a contradiction.

- <u>K is closed</u> suppose $x \notin K$ but that x is an adherent point, with say $a_n \to x$. But a_n has a subsequence a_{n_i} converging to $y \in K$, and $a_{n_i} \to x$, so $x = y \in K$ a contradiction.

This completes the proof $\square$.

We will adopt the property of $[a, b]$ given by the theorem as our definition of <u>compactness</u> in an arbitrary metric space. But first we must define convergence in this generality.

<u>Def^n</u> Let (X, d) be a metric space and $(x_n)_{n=0}^{\infty}$ a sequence in X. Then $(x_n)_{n=0}^{\infty}$ <u>converges</u> to $x \in X$, written $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, if

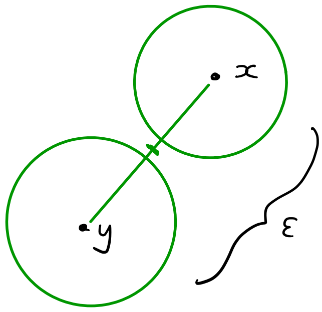
$$\forall \varepsilon > 0 \; \exists N > 0 \; \forall n \in \mathbb{N} \, (n \geq N \Rightarrow d(x_n, x) < \varepsilon).$$

$$(\text{i.e. } \forall \varepsilon > 0 \; \exists N > 0 \, (\{a_n\}_{n \geq N} \subseteq B_\varepsilon(x)))$$

<u>Lemma L8-3</u> If $(x_n)_{n=0}^{\infty}$ has a limit, it is unique.

<u>Proof</u> Suppose $x_n \longrightarrow x$ and $x_n \longrightarrow y$ and set $\varepsilon = d(x,y)$. If $\varepsilon > 0$ then $\varepsilon/2 > 0$ and we may find N with $d(x_n, x) < \varepsilon/2$, $d(x_n, y) < \varepsilon/2$ for all $n \geqslant N$. But then

$$\begin{aligned} d(x,y) &\leqslant d(x, x_n) + d(x_n, y) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$



a contradiction. This proves $\varepsilon = 0$, and thus $x = y$. $\square$

<u>Lemma L8-4</u> A function $f : (X, d) \longrightarrow (Y, d)$ is continuous if and only if whenever $x_n \longrightarrow x$ in X we have $f(x_n) \longrightarrow f(x)$ in Y.

<u>Proof</u> Suppose f is continuous and that $x_n \rightarrow x$. Let $\varepsilon > 0$ be given. The set $f^{-1} B_\varepsilon(fx)$ is open, so $B_\delta(x) \subseteq f^{-1} B_\varepsilon(fx)$ for some δ. Let N be s.t. $\{x_n\}_{n \geqslant N} \subseteq B_\delta(x)$. Then

$$\{ f(x_n) \}_{n \geqslant N} \subseteq f(B_\delta(x)) \subseteq B_\varepsilon(fx)$$

which proves $f(x_n) \rightarrow f(x)$. Now suppose f has the stated property. To prove f is continuous it suffices by Lemma L6-4 to prove

$$\forall x \in X \; \forall \varepsilon > 0 \; \exists \delta > 0 \, \big(y \in B_\delta(x) \Rightarrow f(y) \in B_\varepsilon(f(x)) \big)$$

Suppose otherwise. Then there exists $x \in X$, $\varepsilon > 0$ s.t. $\forall \delta$ $B_\delta(x) \nsubseteq f^{-1} B_\varepsilon(fx)$. Choose for each $n \geqslant 1$ an element

$$x_n \in B_{2^{-n}}(x) \setminus f^{-1} B_\varepsilon(fx) \qquad (\text{note } x_n \neq x)$$

Then $x_n \longrightarrow x$ and hence by hypothesis $f(x_n) \longrightarrow f(x)$, which means in particular there exists N with $d(f(x_n), f(x)) < \varepsilon$ whenever $n \geq N$. But then $f(x_n) \in B_\varepsilon(fx)$ which is a contradiction. $\square$

It is useful to employ this result syntactically as follows:

$$f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n),$$

i.e. continuous functions <u>commute with limits</u>.

<u>Def^n</u> A metric space (X,d) is <u>sequentially compact</u> if every sequence in X has a convergent subsequence. A subset $K \subseteq X$ is called sequentially compact if the metric space $(K, d|_{K\times K})$ is sequentially compact.
(the empty set is allowed, and it is sequentially compact)

<u>Example L8-2</u> By Theorem L8-2 a subset of $\mathbb{R}$ is closed and bounded iff. it is sequentially compact (e.g. $[a,b]$).

Next we make good on the claim that compactness is what is "responsible" for the extreme value theorem.

<u>Proposition L8-5</u> If $f:(X,d) \longrightarrow (Y,d)$ is continuous and $K \subseteq X$ is sequentially compact then $f(K) \subseteq Y$ is sequentially compact.

<u>Proof</u> Suppose K is compact, and let $(y_n)_{n=0}^{\infty}$ be a sequence in $f(K)$. Choose for each $n \geq 0$ an element $x_n \in K$ with $y_n = f(x_n)$. Then x_n has a convergent subsequence x_{n_i} with $\lim_{i\to\infty} x_{n_i} \in K$. Then $y_{n_i} = f(x_{n_i})$ is a subsequence with

$$\lim_{i\to\infty} y_{n_i} = \lim_{i\to\infty} f(x_{n_i})$$

$$= f(\lim_{i\to\infty} x_{n_i}) \in f(K),$$

which completes the proof. □

In particular if X is a sequentially compact metric space and $f: X \longrightarrow \mathbb{R}$ is continuous then $f(X) \subseteq \mathbb{R}$ is sequentially compact, hence closed and bounded. Moreover f attains its minimum and maximum:

<u>Corollary L8-6</u> Let f be a continuous real-valued function on a nonempty sequentially compact metric space (X,d). Then there exist $c,d \in X$ s.t.

$$f(c) \geqslant f(x) \geqslant f(d) \qquad \forall x \in X.$$

<u>Proof</u> Since $f(X)$ is bounded there is a least upper bound λ. This is an adherent point of $f(X)$: if not, there would exist $\varepsilon > 0$ with $B_\varepsilon(\lambda) \cap f(X)$ empty, and $\lambda - \varepsilon/2$ would be an upper bound for $f(X)$, which is a contradiction. But then since $f(X)$ is closed we must have $\lambda \in f(X)$, say $\lambda = f(c)$. Then $f(c) \geqslant f(x)$ for all $x \in X$ as claimed. The other claim is similar. □

<u>Def^n</u> A subset $Y \subseteq X$ of a metric space (X,d) is <u>bounded</u> if there exists $x \in X$ and $\varepsilon > 0$ with $Y \subseteq B_\varepsilon(x)$.

<u>Exercise L8-1</u> (i) Prove Y is bounded if and only if the set $\{d(x,y) \mid x,y \in Y\} \subseteq \mathbb{R}$ is bounded above.

(ii) Prove $Y \subseteq \mathbb{R}$ is bounded in this sense iff. it is bounded in the sense of p. 2.

<u>Exercise L8-2</u> Prove that if $K \subseteq X$ is sequentially compact then it is closed and bounded in X (<u>Hint</u>: recycle the proof of Theorem L8-2).

<u>Exercise L8-3</u> (i) Prove that if (X, d_X) and (Y, d_Y) are isometric then (X, d_X) is sequentially compact iff. (Y, d_Y) is so.
(your proof should not be via (ii) below, i.e. write it in terms of sequences and convergence)

(ii) Prove that if (X, d_X), (Y, d_Y) are metric spaces and the associated <u>topological</u> spaces $(X, \mathcal{T}_{d_X})$, $(Y, \mathcal{T}_{d_Y})$ are homeomorphic then (X, d_X) is sequentially compact iff. (Y, d_Y) is so.
(This is a hint that the proper level of analysis for compactness is the topology rather than the metric. You should prove this directly, i.e. without using the notion of compactness for topological spaces in Lecture 9).

<u>Exercise L8-4</u> Prove any bounded subset of $\mathbb{R}^n$ is contained in $[a,b]^n$ for some $a, b \in \mathbb{R}$.

<u>Exercise L8-5</u> Prove that if $X \subseteq \mathbb{R}$ is <u>not</u> sequentially compact, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ which is not bounded (i.e. $f(X) \subseteq \mathbb{R}$ is not bounded).

<u>Exercise L8-6</u> If (X, d) is sequentially compact and $Y \subseteq X$ is closed then Y is sequentially compact.

Exercise L8-7 Let (X, d) be a metric space and $Y \subseteq X$ a subset.

(i) Prove that the following are equivalent for a point $x \in X$

(a) there is a sequence $(y_n)_{n=0}^{\infty}$ in Y converging to x.
(b) $\forall \varepsilon > 0 \; \exists y \in Y \, (\, d(x, y) < \varepsilon \,)$.

We call such a point x an adherent point of Y.

(ii) Prove that Y is closed in X iff. it contains all of its adherent points.