Lecture 9: Compactness II

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In today's lecture we prove that compactness is a purely topological property, and in so doing, we establish the appropriate generalisation to arbitrary topological spaces. The generalisation is phrased in terms of <u>covers</u>.

<u>Def^n</u> Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C} = \{U_i\}_{i \in I}$ an indexed family of open sets. We say $\mathcal{C}$ <u>covers</u> X (and we call $\mathcal{C}$ an <u>open cover</u>) if (often just "cover")

$$X = \bigcup_{i \in I} U_i.$$

We say $\mathcal{C}$ covers a subset $Y \subseteq X$ if $\{U_i \cap Y\}_{i \in I}$ covers Y, i.e. if $Y \subseteq \bigcup_{i \in I} U_i$. The cover $\mathcal{C}$ is <u>finite</u> if I is finite (I may be empty, in which case $\bigcup_{i \in I} U_i = \emptyset$, so the empty set <u>is</u> a cover of the empty space). A <u>subcover</u> of $\mathcal{C}$ is an indexed set $\{U_j\}_{j \in J}$ for $J \subseteq I$, which is a cover in its own right.

<u>Remark</u> An indexed family of sets is <u>not</u> just a set of sets. That is, the objects $\{U_i\}_{i \in I}$ and $\{U_i \mid i \in I\}$ are different things. In the former we have a set U_i assigned to each $i \in I$ and if $U_i = U_j$ for $i \neq j$ then so be it; in the latter case we just accumulate all the U_i and if $U_i = U_j$ the two sets are identified. For example, if

$$U_1 = \{1, 2, 3\}, \quad U_2 = \{1, 3\}, \quad U_3 = \{1, 2, 3\}$$

then

$$\{U_i \mid i \in \{1,2,3\}\} = \{U_1, U_2, U_3\} = \{\{1,2,3\}, \{1,3\}\}. \qquad (1.1)$$

whereas $\{U_i\}_{i\in\{1,2,3\}}$ actually denotes the <u>function</u>

$$f : \{1,2,3\} \longrightarrow \{U_1, U_2, U_3\}$$
$$f(1) = U_1, \quad f(2) = U_2, \quad f(3) = U_3.$$

If you wish, this function is identified as a set with its graph, which is

$$f = \{(1,\{1,2,3\}), (2,\{1,3\}), (3,\{1,2,3\})\}. \qquad (2.1)$$

So, compare (1.1), (2.1). This is a bit of a technical point, but it is worth getting it straight. A good reference for this kind of thing is T. Tao's "Analysis Vol. I" Chapter 3. Obviously any set X <u>gives rise</u> to an indexed set $\{x\}_{x\in X}$ which is just the identity function $X \to X$. So if we say "the set $Q \subseteq \mathcal{T}$ is a cover" we mean the associated indexed set is a cover, i.e. $X = \bigcup Q$.

<u>Example L9-1</u> (i) $\{X\}$ is a cover of X.

(ii) A basis $\mathcal{B}$ for the topology is a cover.

(iii) $\{(-n,n)\}_{n\geq 1}$ is a cover of $\mathbb{R}$, but not a basis, and $\{(-n,n)\}_{n \text{ even}}$ is a subcover (note that $\{(-n-\frac{1}{2}, n+\frac{1}{2})\}_{n\geq 1}$ is <u>not</u> a subcover. The relevant point is an inclusion of <u>index sets</u>, not of open sets).

<u>Def^n</u> A topological space X is <u>compact</u> if every cover of X has a finite subcover.

<u>Example L9-2</u> $\mathbb{R}$ is not compact, as the cover $\{(-n,n)\}_{n\geq 1}$ has no finite subcover.

<u>Exercise L9-1</u> Prove that $(X, \mathcal{T})$ is compact if and only if for every subset $Q \subseteq \mathcal{T}$ with $X = \bigcup Q$ there exists a finite subset $Q' \subseteq Q$ with $X = \bigcup Q'$ (so it does not really matter if we view covers as "indexed families of open sets" or just subsets of $\mathcal{T}$).

<u>Exercise L9-2</u> Prove that if $(x_n)_{n=0}^{\infty}$ is a sequence in a metric space (X, d) converging to x, then any subsequence also converges to x.

<u>Lemma L9-0</u> Let $\mathcal{B}$ be a basis for a topological space X. Then X is compact if and only if every open cover consisting of sets in $\mathcal{B}$ has a finite subcover.

<u>Proof</u> One direction is clear. Suppose every open cover in $\mathcal{B}$ has a finite subcover and let $\{U_i\}_{i \in I}$ be any open cover. For each $i \in I$ we may write $U_i = \bigcup_{j \in J_i} B_j$ for some set J_i and $B_j \in \mathcal{B}$. But then with $J = \bigcup_{i \in I} J_i$

$$X = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{j \in J_i} B_j = \bigcup_{j \in J} B_j$$

and by hypothesis there is $J' \subseteq J$ finite s.t. $X = \bigcup_{j \in J'} B_j$. We may assume $J_i \cap J_{i'} = \phi$ if $i \neq i'$, then $I' = \{ i \in I \mid J_i \cap J' \neq \phi \}$ is finite and for every $j \in J'$ there exists $i \in I'$ with $j \in J_i$ and thus $B_j \subseteq U_i$, so that

$$\bigcup_{i \in I'} U_i \supseteq \bigcup_{j \in J'} B_j = X$$

which proves $\{U_i\}_{i \in I'}$ is a cover of X and completes the proof. $\square$

<u>Exercise L9-3</u> Prove that in a metric space (X,d) any convergent sequence $(x_n)_{n=0}^{\infty}$ is <u>Cauchy</u>, i.e. $\forall \varepsilon > 0 \; \exists N \; \forall m,n \geqslant N \;\; d(x_m, x_n) < \varepsilon$.

The following argument is a slightly modified form of J.R.Munkres "Topology" §28.

<u>Lemma L9-1</u> Let (X,d) be a sequentially compact metric space. Given $\varepsilon > 0$ there exists some finite set $x_1, \ldots, x_n \in X$ such that $\{ B_\varepsilon(x_i) \}_{i=1}^{n}$ covers X.

<u>Proof</u> Given $\varepsilon > 0$ suppose no such finite cover of ε-balls existed. Choose some $x_1 \in X$. Then $B_\varepsilon(x_1) \neq X$ so we may choose $x_2 \in X \setminus B_\varepsilon(x_1)$. Since $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \neq X$ we may choose $x_3 \in X \setminus B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ and in this way construct $(x_n)_{n=0}^{\infty}$ with $d(x_n, x_m) \geqslant \varepsilon$ whenever $n \neq m$.
By hypothesis $(x_n)_{n=0}$ contains a convergent subsequence, but this subsequence clearly cannot be Cauchy and so we have a contradiction. $\square$

<u>Theorem L9-2</u> Let (X,d) be a metric space with associated topological space $(X,\mathcal{T})$. Then (X,d) is sequentially compact iff. $(X,\mathcal{T})$ is compact.

<u>Proof</u> Suppose (X,d) is sequentially compact and that $\mathcal{C} = \{U_i\}_{i \in I}$ is an open cover. We need to show $\mathcal{C}$ contains a finite subcover.

<u>Claim</u> : there is $\delta > 0$ such that for every $y \in X$, there exists an $i \in I$ with $B_\delta(y) \subseteq U_i$.

<u>Proof of Claim</u> : suppose not. Then for n an integer, there exists $y_n \in X$ with $B_{1/n}(y_n) \nsubseteq U_i$ for any $i \in I$. The sequence $(y_n)_{n=0}^{\infty}$ contains a convergent subsequence, say $y_{n_t} \longrightarrow y$.

We have $y \in U_i$ for some i, and thus $B_\varepsilon(y) \subseteq U_i$ for some $\varepsilon > 0$. There exists N s.t. for $t \geq N$, $y_{n_t} \in B_{\varepsilon/2}(y)$. But for $t >> 0$ we have $1/n_t < \varepsilon/2$ and thus for $z \in X$ if $d(z, y_{n_t}) < 1/n_t$ then

$$\begin{aligned} d(z,y) &\leq d(z, y_{n_t}) + d(y_{n_t}, y) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

which implies

$$B_{1/n_t}(y_{n_t}) \subseteq B_\varepsilon(y) \subseteq U_i$$

a contradiction. $\square$

Let $\delta > 0$ be as in the claim and use Lemma L9-1 with $\varepsilon = \delta$ to produce a set of points $x_1, \dots, x_r$ s.t. $\{B_\delta(x_i)\}_{i=1}^r$ cover X. But each of these balls lies in some U_i, say $U_{i_1}, \dots, U_{i_r}$ respectively. Then

$$\begin{aligned} X &= B_\delta(x_1) \cup \cdots \cup B_\delta(x_r) \\ &\subseteq U_{i_1} \cup \cdots \cup U_{i_r} \end{aligned}$$

so $\{U_{i_1}, \dots, U_{i_r}\}$ is a finite subcover.

For the converse, suppose $(X,\mathcal{T})$ is compact and that $(x_n)_{n=0}^{\infty}$ is any sequence in X. Suppose it contains no convergent subsequence and set $A = \{x_n \mid n \geq 0\}$. We claim A is closed. If $x \notin A$ is an adherent point of A then we may find n_1 with $d(x_{n_1}, x) < 1$, and $n_2 > n_1$ with $d(x_{n_2}, x) < 1/2$ and in this way find a subsequence converging to x, a contradiction.

So A is closed. Now, as no subsequence of $(x_n)_{n=0}^{\infty}$ converges to any particular x_k, given $k \geq 0$ we may find $\varepsilon_k > 0$ such that

$$\{n \geq 0 \mid x_n \in B_{\varepsilon_k}(x_k)\}$$

is finite (if for all $\varepsilon > 0$ this set were infinite, taking $\varepsilon = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$ we would construct a subsequence converging to x_k as above). But as A is closed,

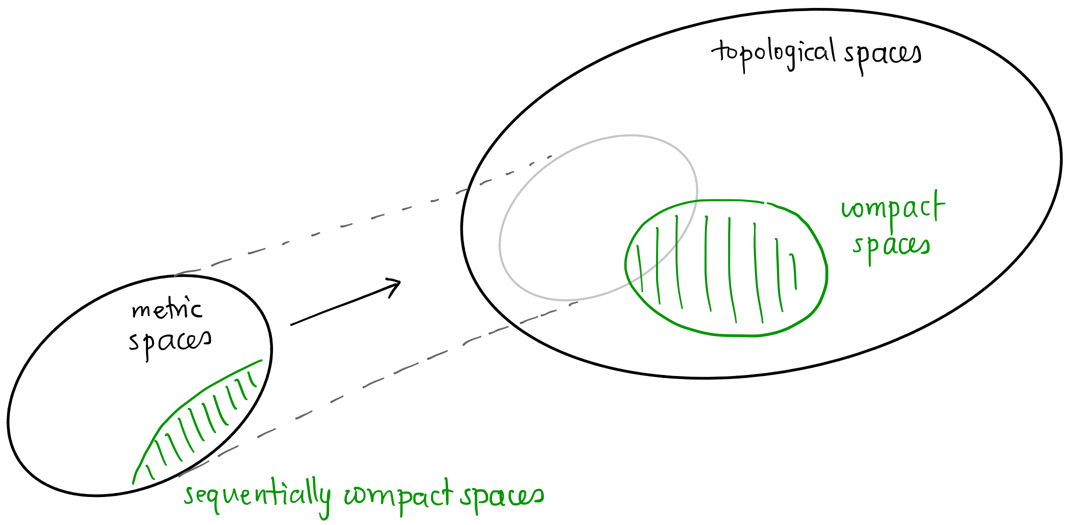
$$\{X \setminus A\} \cup \{B_{\varepsilon_k}(x_k)\}_{k=0}^{\infty}$$

is an open cover of X, which must have have some finite subcover

$$\{X \setminus A, B_{\varepsilon_{k_1}}(x_{k_1}), \ldots, B_{\varepsilon_{k_r}}(x_{k_r})\}.$$

But then some $B_{\varepsilon_{k_i}}(x_{k_i})$ must contain x_n for infinitely many n! This contradiction shows that, indeed, $(x_n)_{n=0}^{\infty}$ must contain a convergent subsequence. $\square$

The theorem shows sequential compactness is actually a property of the topology. Henceforth when we say a metric space (X,d) is <u>compact</u> we mean $(X, \mathcal{T}_d)$ is compact, or equivalently, that (X,d) is sequentially compact (but we will basically stop using this terminology).



Now that we have generalised the notion of sequential compactness to compactness for arbitrary topological spaces, we can proceed to generalise various facts from Lecture 8, in particular Proposition L8-5. Once again given a space X, a subset K is is called <u>compact</u> if $(K, \mathcal{T}_X|_K)$ is compact.

<u>Exercise L9-4</u> Prove a subset K of a topological space X is compact iff. for every indexed family of open sets $\{U_i\}_{i\in I}$ with $K \subseteq \bigcup_{i\in I} U_i$ there is finite $I' \subseteq I$ with $K \subseteq \bigcup_{i\in I'} U_i$.

<u>Proposition L9-3</u> If $f: X \longrightarrow Y$ is continuous and $K \subseteq X$ is compact then $f(K) \subseteq Y$ is compact.

<u>Proof</u> Suppose $f(K) \subseteq \bigcup_{i\in I} U_i$ for open sets U_i. Then

$$K \subseteq f^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} f^{-1}(U_i)$$

and so since K is compact there is $I' \subseteq I$ finite with $K \subseteq \bigcup_{i\in I'} f^{-1}(U_i)$. But then $f(K) \subseteq \bigcup_{i\in I'} U_i$ so we are done. $\square$

The Extreme Value Theorem also holds in this generality:

<u>Corollary L9-4</u> Let f be a continuous real-valued function on a nonempty compact topological space X. Then there exist $c, d \in X$ s.t.

$$f(c) \geqslant f(x) \geqslant f(d) \qquad \forall x \in X.$$

<u>Proof</u> Same as Corollary L8-6, using Proposition L9-3. $\square$

<u>Exercise L9-5</u> Every closed subspace of a compact topological space is compact.

<u>Exercise L9-6</u> When is a discrete space (i.e. a topological space with the discrete topology) compact?

<u>Exercise L9-7</u> Let (X,d) be a metric space and $K \subseteq X$ a nonempty compact subset and $x \in X \setminus K$. Set

$$\lambda_x := \inf\{ d(k,x) \mid k \in K \}.$$

Prove that there exists $k_0 \in K$ with

$$d(k_0, x) = \lambda_x.$$