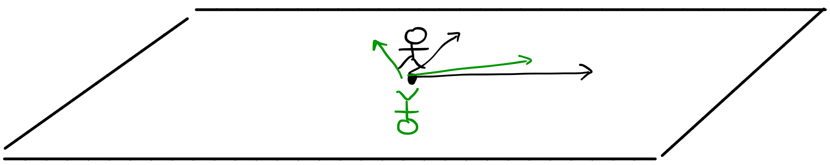
MAST30026 Tutorial 1 (2019)

Consider two observers O_1, O_2 who are measuring points in the same abstract plane X from different sides (imagine a physical sheet) but at the same point $\underline{0}$, with their axes rotated by some angle relative to one another



Exercise L3-6 Prove that there is an element $F \in O(2)$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "m_2"'] \arrow[dr, "m_1"] & \\ \mathbb{R}^2 \arrow[rr, "F"'] & & \mathbb{R}^2 \end{tikzcd}$$

commutes, i.e. a fixed orthogonal matrix which converts O_2's measurements to m_1's measurements.

Definition Given two ordered bases $\mathcal{B} = (\underline{b}_1, \ldots, \underline{b}_n)$, $\mathcal{C} = (\underline{c}_1, \ldots, \underline{c}_n)$ for a finite-dimensional vector space V, we say $\mathcal{B}, \mathcal{C}$ have the same orientation and write $\mathcal{B} \sim \mathcal{C}$ if

$$\det\left([\mathrm{Id}]_{\mathcal{B}}^{\mathcal{C}} \right) > 0$$

where $[\mathrm{Id}]_{\mathcal{B}}^{\mathcal{C}} = \left([\underline{b}_1]_{\mathcal{C}}, \ldots, [\underline{b}_n]_{\mathcal{C}} \right)$ is the change of basis matrix.

What are some obvious examples of ordered bases with <u>different</u> orientations?

$\boxed{Q1}$ (i) $\sim$ is an equivalence relation on the set $\mathcal{F}$ of ordered bases of V.

(ii) there are precisely two equivalence classes, i.e. $\mathcal{F}/\sim$ has two elements.

<u>Exercise L3-5</u> (i) Let $F: V \longrightarrow V$ be an invertible linear operator on a finite-dimensional vector space. Prove that precisely one of the following two possibilities is realised:

$$(\mathrm{I}) \quad \forall \beta \left(F(\beta) \sim \beta \right)$$

(β ranges over all ordered bases)

$$(\mathrm{II}) \quad \forall \beta \left(F(\beta) \nsim \beta \right)$$

where $F(\beta)$ denotes $(F(\underline{b}_1), \dots, F(\underline{b}_n))$ if $\beta = (\underline{b}_1, \dots, \underline{b}_n)$. In the first case we say F is <u>orientation preserving</u> and in the latter case we say F is <u>orientation reversing</u>.

(ii) Prove that F is orientation preserving iff. $\det(F) > 0$, and orientation reversing iff. $\det(F) < 0$.

With this language we can clarify the comments about orientation in lectures. Two observers in the plane, whose coordinate systems may differ by a rotation <u>and</u> possibly "being on the other side of the plane", have their measurements related by $F \in O(2)$ by Ex. L3-6. They "agree on clockwise" iff. F is orientation preserving which is iff. $F \in SO(2) \subseteq O(2)$.