Tutorial #1 (2020)

Cartesian product Given sets X, Y you know

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}.$$

But what is an ordered pair (x,y)? It isn't some axiom of set theory that these things exist (it is usually an axiom in type theory, but this is a different story). The ordered pair is defined from more primitive elements. Usually we use the Kuratowski pair

$$(x,y) := \{\{x\}, \{x,y\}\}.$$

Q1 Prove $(x,y) = (a,b) \iff (x = a \wedge y = b)$. Note that sets are equal iff. they have the same elements.

Well what about $X \times Y \times Z$? Should we invent a "Kuratowski triple"? No. Let us make a general definition: suppose we have a collection of sets $\{X_i\}_{i \in I}$ indexed by a set I. We want elements of $\prod_{i \in I} X_i$ to be I-indexed families of elements $x_i \in X_i$, one for each $i \in I$. That is

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i \in I\}$$

But what does $(x_i)_{i \in I}$ really mean? What set is it?

Let us do an easy case: $I = \{1,2,3\}$. Then an element of $\prod_{i \in I} X_i$ is just an ordered triple (x_1, x_2, x_3) with $x_i \in X_i$. This we may realise as

$$(x_1, x_2, x_3) := \{(1, x_1), (2, x_2), (3, x_3)\}$$

Then more generally,

$$(x_i)_{i\in I} := \{ (i, x_i) \mid i \in I \} \qquad (2.1)$$

$\boxed{Q2}$ Prove $(x_i)_{i\in I} = (y_i)_{i\in I} \iff x_i = y_i$ for all $i \in I$.

Then $\prod_{i\in I} X_i$ is the set of all sets like (2.1) above, i.e. sets S where

(i) every element is a pair, the first entry of which lies in I

(ii) for each $i \in I$ there is a unique pair in S beginning with i.

(iii) if $(i, x) \in S$ then $x \in X_i$.

We call this set the Cartesian product.

$\boxed{Q3}$ Prove that $\prod_{i\in\{1,2\}} X_i$ is bijective to $X_1 \times X_2$ (defined by Kuratowski pairs). Are these literally the same set?

Disjoint union How do I form a set S which contains two copies of a given set, say the interval $X = [0,1]$? well

$$X \cup Y = \{ x \mid x \in X \text{ or } x \in Y \}$$

so $[0,1] \cup [0,1] = [0,1]$. The ordinary union won't work.
Idea we can "colour" one copy of X in some arbitrary way, i.e. $X' := \{ (0,x) \mid x \in X \}$. This is bijective to, but distinct from, X.
Then $S := X \cup X'$ contains two distinct copies of X.

We call S the <u>disjoint union</u> of two copies of X and write $S = X \amalg X$.
More generally

$$X \amalg Y := (\{0\} \times X) \cup (\{1\} \times Y)$$

More generally, given a collection $\{X_i\}_{i \in I}$ we "colour" X_i by i in order to separate each X_i from X_j, <u>even if</u> as sets $X_i = X_j$, to form the <u>disjoint union</u>

$$\coprod_{i \in I} X_i := \bigcup_{i \in I} \{i\} \times X_i$$
$$= \{ (i, x) \mid i \in I \text{ and } x \in X_i \}.$$

$\boxed{Q4}$ Give injective functions $g_i : X_i \longrightarrow \coprod_{i \in I} X_i$ s.t.

- $i \neq j \Rightarrow g_i(X_i) \cap g_j(X_j) = \emptyset$
- $\bigcup_{i \in I} g_i(X_i) = \coprod_{i \in I} X_i$.

$\boxed{Q5}$ (i) Identify $\prod_{i \in I} X_i$ with a subset of the set of functions

$$I \longrightarrow \bigcup_{i \in I} X_i .$$

(ii) Prove that given subsets U_i, V_i of X_i, $\left(\coprod_{i \in I} U_i\right) \cap \left(\coprod_{i \in I} V_i\right)$ is equal to $\coprod_{i \in I} (U_i \cap V_i)$ as subsets of the disjoint union of the X_i.

$\boxed{Q6}$ Let X be a set. Prove that if $E_i \subseteq X \times X$ is an equivalence relation then so is $\bigcap_{i \in I} E_i$. Given $Q \subseteq X \times X$ <u>any</u> set, let

$$E = \bigcap \{ Y \subseteq X \times X \mid Y \text{ is an equiv. rel}^{N} \ \& \ Y \supseteq Q \}$$

This is the equivalence relation <u>generated by Q</u>.

$\boxed{Q7}$ What is the equivalence relation on $\{1, 2, 3, 4\}$ generated by the pairs $Q = \{(1,2), (4,3)\}$