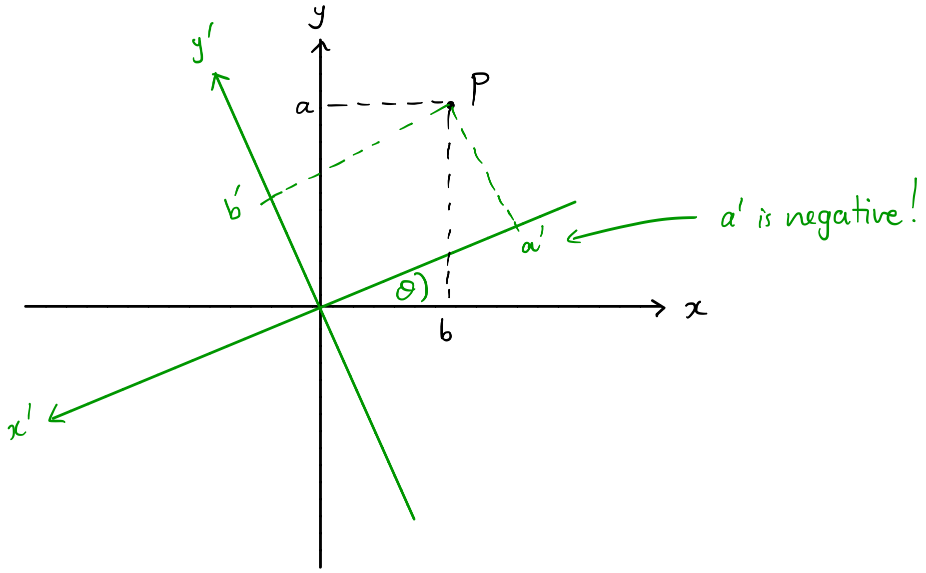
Solutions

$\boxed{\text{Ex. L3-6}}$ Consider the diagram



We know from our earlier calculations that

$$m_1(p) = \begin{pmatrix} a \\ b \end{pmatrix} = R_\theta \begin{pmatrix} -a' \\ b' \end{pmatrix} = R_\theta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = R_\theta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} m_2(p)$$

Now $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is orthogonal, so $R_\theta S \in O(2)$ and we are done. How can we write $R_\theta S = R_\psi T$ for some ψ?

$\boxed{\text{Q1}}$ (i) We must show $\sim$ is reflexive, symmetric and transitive.

- reflexive $[Id]^\beta_\beta = I_n$ so $\det([Id]^\beta_\mathcal{C}) = 1$ and $\beta \sim \beta$
- symmetric $[Id]^\beta_\mathcal{C} = \left([Id]^\mathcal{C}_\beta\right)^{-1}$ so $\det([Id]^\beta_\mathcal{C}) = \det([Id]^\mathcal{C}_\beta)$ and so clearly $\beta \sim \mathcal{C}$ iff. $\mathcal{C} \sim \beta$

(iii) <u>transitivity</u> if $\beta \sim \mathcal{C}$ and $\mathcal{C} \sim \mathcal{D}$ then

$$[Id]_{\beta}^{\mathcal{D}} = [Id]_{\mathcal{C}}^{\mathcal{D}} \, [Id]_{\beta}^{\mathcal{C}}$$

and hence

$$\det([Id]_{\beta}^{\mathcal{D}}) = \det([Id]_{\mathcal{C}}^{\mathcal{D}}) \cdot \det([Id]_{\beta}^{\mathcal{C}})$$

since the product of positive numbers is positive, $\det([Id]_{\beta}^{\mathcal{D}}) > 0$ and so $\beta \sim \mathcal{D}$.

(ii) If $\beta \nsim \mathcal{C}$ then $\beta' \sim \mathcal{C}$ where β' is obtained by multiplying by -1 one of basis vectors in β, say $\beta' = (-\underline{b}_1, \underline{b}_2, \dots)$, since

$$\det([Id]_{\beta}^{\mathcal{C}}) = \det([\underline{b}_1]_{\mathcal{C}}, [\underline{b}_2]_{\mathcal{C}}, \dots)$$

$$= -\det([-\underline{b}_1]_{\mathcal{C}}, [\underline{b}_2]_{\mathcal{C}}, \dots)$$

$$= -\det([Id]_{\beta'}^{\mathcal{C}})$$

So there are <u>at most</u> two equivalence classes. To show there are at least two, let $\beta = (\underline{b}_1, \dots, \underline{b}_n)$ be any basis and set $\beta' = (-\underline{b}_1, \underline{b}_2, \dots)$ then $\beta' \nsim \beta$.

$\boxed{\text{Ex L3-5}}$ Note that for any ordered basis

$$[Id]^{\beta}_{F(\beta)} = \left([F(b_1)]_\beta, \dots, [F(b_n)]_\beta \right) = [F]^{\beta}_{\beta}$$

We know $\det(F) = \det([F]^{\beta}_{\beta})$ is independent of β, so

$$(\exists \beta \ F(\beta) \sim \beta) \Longrightarrow (\exists \beta \ \det([Id]^{\beta}_{F(\beta)}) > 0)$$

$$\Rightarrow \det(F) > 0$$

$$\Rightarrow (\forall \beta \ F(\beta) \sim \beta).$$

Now let F be fixed and choose any ordered basis β. There are two cases, by Q1 (ii)

(I) $F(\beta) \sim \beta$, in which case $(\forall \mathcal{C} \ F(\mathcal{C}) \sim \mathcal{C})$

(I) $F(\beta) \nsim \beta$, in which case $(\forall \mathcal{C} \ F(\mathcal{C}) \nsim \mathcal{C})$.

(ii) is clear from the proof of (i).