Tutorial 1 solutions

Q1/ A permutation σ on n letters is a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$.
Associated to an ordered sequence $a_1, \dots, a_k$ in $\{1, \dots, n\}$ of length $k \geq 1$ with $a_i \neq a_j$ for $i \neq j$ is a permutation σ defined by

$$\sigma(x) = \begin{cases} a_2 & \text{if } x = a_1 \\ a_3 & \text{if } x = a_2 \\ \vdots & \\ a_k & \text{if } x = a_{k-1} \\ a_1 & \text{if } x = a_k \\ x & \text{otherwise} \end{cases}$$

we denote this by $(a_1 \cdots a_k)$. Such a permutation is called a <u>cycle</u>.
A <u>transposition</u> is a cycle $(a_1\, a_2)$ of length two.

<u>Claim</u> Any permutation can be written as a product of cycles.

<u>Proof</u> Suppose not, for a contradiction, and let $n \geq 2$ be the smallest integer such that $\sigma \in S_n$ exists which is not a product of cycles
Consider the sequence

$$1,\ \sigma(1),\ \sigma^2(1), \dots,\ \sigma^k(1) \qquad 0 \leq k \leq n-1$$

where k is the least value such that $\sigma^{k+1}(1) = 1$.
Set $\Omega = \{1, \dots, n\} \setminus \{1, \sigma(1), \dots, \sigma^k(1)\}$. Then $\sigma|\Omega$ is a bijection on Ω and so by minimality of σ it is a product of cycles, hence also $\sigma = \sigma|\Omega\,(1\ \sigma(1)\ \cdots\ \sigma^k(1))$ is a product of cycles, which is a contradiction. This proves the claim. $\square$

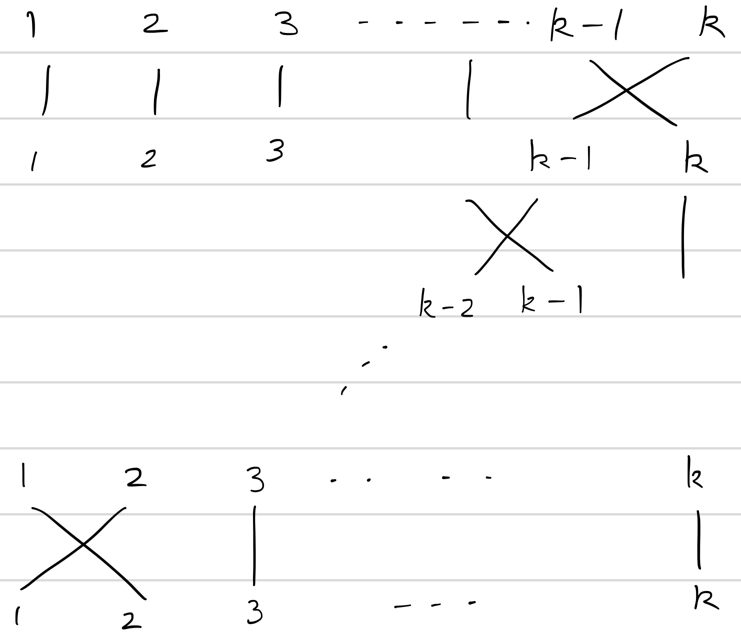
<u>Claim</u> Every permutation is a product of transpositions.

<u>Proof</u> In light of the above, it suffices to prove every cycle $(a_1 \cdots a_k)$ is a product of transpositions. But we can just exhibit this directly:

$$(a_1 \cdots a_k) = (a_1\ a_2) \cdots (a_{k-1}\ a_{k-2})(a_k\ a_{k-1}) \qquad (*)$$

as claimed. □

To understand $(*)$ first observe it is enough to understand $(1\ 2 \cdots k)$, and



does the trick.

$\boxed{Q2}$ (iii), (iv) are easy so we omit them.

(i) $(I_n)_{ij} = \delta_{ij}$ so

$$\det(I_n) = \sum_{\sigma} (-1)^{|\sigma|} (I_n)_{1\sigma(1)} \cdots (I_n)_{n\sigma(n)}$$

$$= \sum_{\sigma} (-1)^{|\sigma|} \delta_{1\sigma(1)} \cdots \delta_{n\sigma(n)}$$

The only permutation contibuting to this sum is $\sigma = id$, so

$$= \delta_{11} \cdots \delta_{nn} = 1.$$

(ii) $$\det(XY) = \sum_{\sigma} (-1)^{|\sigma|} (XY)_{1\sigma(1)} \cdots (XY)_{n\sigma(n)}$$

$$= \sum_{\sigma} (-1)^{|\sigma|} \sum_{i_1} x_{1i_1} y_{i_1\sigma(1)} \cdots \sum_{i_n} x_{ni_n} y_{i_n\sigma(n)}$$

$$= \sum_{i_1 \cdots i_n} \sum_{\sigma} (-1)^{|\sigma|} x_{1i_1} x_{2i_2} \cdots x_{ni_n} \cdot y_{i_1\sigma(1)} \cdots y_{i_n\sigma(n)}$$

Observe that if $\underline{i} = (i_1, \ldots, i_n)$ has a repeated index, say $i_a = i_b$ with $a < b$, then with $\tau = \sigma(a\ b)$

$$x_{1i_1} \cdots x_{ai_a} \cdots x_{bi_b} \cdots x_{ni_n} \cdot y_{i_1} \cdots y_{i_a\sigma(a)} \cdots y_{i_b\sigma(b)} \cdots y_{i_n\sigma(n)}$$

$$= x_{1i_1} \cdots x_{ai_b} \cdots x_{bi_a} \cdots x_{ni_n} \cdot y_{i_1\sigma(1)} \cdots y_{i_b\sigma(a)} \cdots y_{i_a\sigma(b)} \cdots y_{i_n\sigma(n)}$$

$$= x_{1i_1} \cdots x_{ai_b} \cdots x_{bi_a} \cdots x_{ni_n} \cdot y_{i_1\tau(1)} \cdots y_{i_b\tau(b)} \cdots y_{i_a\tau(a)} \cdots y_{i_n\tau(n)}$$

But with $\underline{i}$ as above this shows

$$\sum_{\sigma} (-1)^{|\sigma|} x_{1i_1} x_{2i_2} \cdots x_{ni_n} \cdot y_{i_1\sigma(1)} \cdots y_{i_n\sigma(n)}$$

is zero, because we can divide S_n into pairs by the equivalence relation $\sigma \sim \tau$ if $\sigma = \tau(a\,b)$ (equivalently $\sigma(a\,b) = \tau$) and each of these pairs add to zero in the sum since $|\sigma(a\,b)| = |\sigma| + 1$, so they have opposite sign.

So, we have shown that in the sum we may restrict to sequences $\underline{i}$ with no repetitions. But such a sequence is a <u>permutation</u> of $\{1, \ldots, n\}$ so

$$\det(XY) = \sum_{\theta \in S_n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} x_{1\theta(1)} \cdots x_{n\theta(n)} \, y_{\theta(1)\sigma(1)} \cdots y_{\theta(n)\sigma(n)}$$

Notice that

$$y_{\theta(1)\sigma(1)} \cdots y_{\theta(n)\sigma(n)} = y_{1\,\sigma\theta^{-1}(1)} \cdots y_{n\,\sigma\theta^{-1}(n)}$$

so this is

$$\det(XY) = \sum_{\theta \in S_n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} x_{1\theta(1)} \cdots x_{n\theta(n)} \cdot y_{1\,\sigma\theta^{-1}(1)} \cdots y_{n\,\sigma\theta^{-1}(n)}$$

But if we fix θ and sum over all σ, then $\{\sigma\theta^{-1}\}_{\sigma \in S_n}$ just enumerates all permutations, so we may as well just replace Σ_σ by an enumeration of these permutations (say $\alpha = \sigma\theta^{-1}$) directly, and replace $|\sigma|$ by $|\alpha\theta|$)

$$= \sum_{\theta, \alpha \in S_n} (-1)^{|\alpha||\theta|} x_{1\theta(1)} \cdots x_{n\theta(n)} \, y_{1\alpha(1)} \cdots y_{n\alpha(n)}$$

$$= \det(X)\det(Y). \quad \square$$

$\boxed{Q3}$ (i) We need to prove that

<u>$\sim$ is reflexive</u> $\beta \sim \beta$ since $[Id_V]_\beta^\beta = I_n$ and $\det(I_n) = 1 > 0$

<u>$\sim$ is symmetric</u> if $\beta \sim \mathcal{C}$ then

$$\det([Id_V]_\mathcal{C}^\beta) = \det([Id_V]_\beta^\mathcal{C})^{-1} > 0$$

since $\det([Id_V]_\beta^\mathcal{C}) > 0$. Hence $\mathcal{C} \sim \beta$.

<u>$\sim$ is transitive</u> suppose $\beta \sim \mathcal{C}$ and $\mathcal{C} \sim \mathcal{D}$. Then $\det([Id_V]_\beta^\mathcal{C}) > 0$ and $\det([Id_V]_\mathcal{C}^\mathcal{D}) > 0$. Hence

$$\det([Id_V]_\beta^\mathcal{D}) = \det([Id_V]_\mathcal{C}^\mathcal{D}[Id_V]_\beta^\mathcal{C})$$
$$= \det([Id_V]_\mathcal{C}^\mathcal{D})\det([Id_V]_\beta^\mathcal{C}) > 0$$

which shows $\beta \sim \mathcal{D}$.

(ii) Choose an arbitrary ordered basis $\beta = (\underline{b}_1, \dots, \underline{b}_n)$ and let $\beta' = (\underline{b}_2, \underline{b}_1, \underline{b}_3, \dots, \underline{b}_n)$. We claim

$$\mathcal{F}/\sim \; = \{[\beta], [\beta']\}.$$

Clearly $\det([Id_V]_\beta^{\beta'}) = -1$ so $\beta \nsim \beta'$. Now let $\mathcal{C}$ be any ordered basis. We have to show $\beta \sim \mathcal{C}$ or $\beta' \sim \mathcal{C}$. Suppose $\beta \nsim \mathcal{C}$, so $\det([Id_V]_\beta^\mathcal{C}) < 0$. Then

$$\det([Id_V]_{\beta'}^\mathcal{C}) = \det([Id_V]_\beta^\mathcal{C}[Id_V]_{\beta'}^\beta) = -\det([Id_V]_\beta^\mathcal{C}) > 0$$

so $\beta' \sim \mathcal{C}$, completing the proof. $\square$