Tutorial 10: Dual spaces

Dual vector spaces and the duals of linear maps will be central objects of study in the remainder of the course, and this tutorial prepares the ground with some of the basics. Throughout k is a field and vector spaces are over k, not necessarily *finite-dimensional*.

<u>Def^n</u> Given vector spaces V, W the set

$$\text{Lin}(V, W) := \{ f : V \longrightarrow W \mid f \text{ is linear} \}$$

is a vector space with the "pointwise" operations defined for $f, g \in \text{Lin}(V, W)$ and $\lambda \in \mathbb{R}$ by

- $(f + g)(v) = f(v) + g(v)$
- $(-f)(v) = -f(v)$
- $0(v) = 0_W$. $\qquad$ (i.e. $0 \in \text{Lin}(V, W)$)
- $(\lambda f)(v) = \lambda \cdot f(v)$.

<u>Def^n</u> The <u>dual space</u> V^* of a vector space V is $V^* := \text{Lin}(V, k)$. We often refer to vectors $\mu \in V^*$ as <u>functionals</u>.

Clearly if U, V, W are vector spaces and $f : U \longrightarrow V$, $g : V \longrightarrow W$ are linear then the functions given below are also linear:

$$\text{Lin}(U, V) \overset{g \circ (-)}{\longrightarrow} \text{Lin}(U, W) \qquad h \longmapsto g \circ h$$

$$\text{Lin}(V, W) \overset{(-) \circ f}{\longrightarrow} \text{Lin}(U, W) \qquad h \longmapsto h \circ f$$

<u>Def^n</u> If $f: U \longrightarrow V$ is linear then the linear transformation

$$V^* = \text{Lin}(V, k) \longrightarrow \text{Lin}(U, k) = U^*$$

$$h \longmapsto h \circ f$$

is denoted $f^*: V^* \longrightarrow U^*$.

[Q1] Prove that $(1_U)^* = 1_{U^*}$ and if $f: U \longrightarrow V$, $g: V \longrightarrow W$ are linear then $f^* \circ g^* = (g \circ f)^*$ as linear maps $W^* \longrightarrow U^*$. We say the dual is a <u>functorial</u> operation. Conclude that if f is an isomorphism then so is f^*.

[Q2] Prove that $(f + g)^* = f^* + g^*$ and $(\lambda f)^* = \lambda \cdot f^*$ so that taking the dual is a linear map $(-)^*: \text{Lin}(V, W) \longrightarrow \text{Lin}(W^*, V^*)$.

Given $A \in M_{m,n}(k)$ let $M_A: k^n \longrightarrow k^m$ be $M_A(x) = Ax$. Then there is an isomorphism of vector spaces

$$M_{m,n}(k) \overset{\cong}{\longrightarrow} \mathcal{L}(k^n, k^m)$$

$$A \longmapsto M_A$$

Suppose that V is <u>finite-dimensional</u> and that $\beta = (v_1, \ldots, v_n)$ is a sequence of vectors in V. Then β is an ordered basis <u>if and only if</u>

$$c_\beta: k^n \longrightarrow V, \qquad c_\beta(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i v_i$$

is an isomorphism of vector spaces (it is always a well-defined linear map).

If W is another finite-dimensional vector space with ordered basis $\mathcal{C} = (w_1, \ldots, w_m)$ then given a linear transformation $F: V \to W$ there is a <u>unique</u> matrix A such that the following diagram commutes:

$$
\begin{array}{ccc}
k^n & \xrightarrow{\ M_A\ } & k^m \\
c_\beta \downarrow \cong & & \cong \downarrow c_\mathcal{C} \\
V & \xrightarrow[\ F\ ]{} & W
\end{array}
\qquad (*)
$$

We call A the <u>matrix of F with respect to $\beta, \mathcal{C}$</u>. Some common notation:

- Given $v \in V$ we write $[v]_\beta := c_\beta^{-1}(v)$ (similarly $[w]_\mathcal{C}$).

- We write $[F]_\beta^\mathcal{C}$ for A. We may calculate from $(*)$ that the ith column of this matrix is

$$
\begin{aligned}
\left(c_\mathcal{C}^{-1} \circ F \circ c_\beta \right)(e_i) &= \left(c_\mathcal{C}^{-1} \circ F \right)(v_i) \\
&= c_\mathcal{C}^{-1}\left(F(v_i) \right) \\
&= \left[F(v_i) \right]_\mathcal{C}.
\end{aligned}
$$

Q3 Use the diagram $(*)$ to show that if $G: W \to U$ is another linear transformation and $\mathcal{D}$ is an ordered basis for U then

$$
[G]_\mathcal{C}^\mathcal{D} \circ [F]_\beta^\mathcal{C} = [G \circ F]_\beta^\mathcal{D}.
$$

Next we observe that a familiar matrix operation, the transpose, is "really" about dual spaces, in the following sense.

<u>Definition</u> If V is a vector space, a function $B : V \times V \longrightarrow \mathbb{R}$ is <u>bilinear</u> if

(i) $B(x+x', y) = B(x,y) + B(x', y)$ $\forall x, x', y \in V$

(ii) $B(x, y+y') = B(x,y) + B(x, y')$ $\forall x, y, y' \in V$

(iii) $B(\lambda x, y) = B(x, \lambda y) = \lambda B(x,y)$ $\forall x, y \in V$ $\forall \lambda \in \mathbb{R}$.

We say B is <u>symmetric</u> if $B(x,y) = B(y,x)$ for all $x, y \in V$.

[Q4] Prove that there is a bijection between bilinear forms on V and linear maps $V \longrightarrow V^*$ where V^* denotes the dual space, where the linear map f_B associated to B is $f_B(x)(y) = B(x,y)$.

$$\left\{ \begin{array}{c} \text{bilinear pairings} \\ V \times V \longrightarrow k \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{linear maps} \\ V \longrightarrow V^* \end{array} \right\}$$

$$B \longmapsto \{ v \mapsto B(v, -) \}$$

[Q5] Suppose V is finite-dimensional. A symmetric bilinear form B on V is called <u>nondegenerate</u> if f_B is an isomorphism $V \overset{\cong}{\longrightarrow} V^*$. Write down the definition of nondegeneracy purely in terms of pairs (x,y) for which $B(x,y) = 0$.

<u>Example</u> The pairing $B : k^n \times k^n \longrightarrow k$, $B(\underline{a}, \underline{b}) = \sum_{i=1}^{n} a_i b_i$ is nondegenerate, and the associated isomorphism is

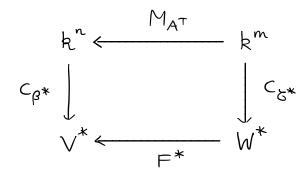$$k^n \overset{\cong}{\longrightarrow} (k^n)^* \qquad\qquad (**)$$

$$e_i \longmapsto e_i^*$$

where $e_i^*(\underline{a}) = a_i$. In particular $(e_1^*, \ldots, e_n^*)$ must be a basis for $(k^n)^*$, called the <u>dual basis</u>.

<u>Defⁿ</u> Let V be a finite-dimensional vector space with ordered basis $\beta = (v_1, \ldots, v_n)$. The <u>dual basis</u> is $\beta^* = (v_1^*, \ldots, v_n^*)$ for V^* is

$$v_i^*(u) = e_i^*([u]_\beta)$$

which reads off the coefficient of v_i in the expansion of u w.r.t. β.

$\boxed{Q6}$ Prove β^* is a basis for V^*.

$\boxed{Q7}$ Suppose V, W, β, G, F, A are as in $(*)$ on p. ③. Prove that

$$k^n \xleftarrow{\quad M_{A^T} \quad} k^m$$

$$\Big\downarrow C_{\beta^*} \qquad\qquad \Big\downarrow C_{G^*}$$

$$V^* \xleftarrow{\quad F^* \quad} W^*$$

commutes, and hence $[F^*]_{G^*}^{\beta^*} = \left([F]_\beta^G\right)^T$, so that taking the transpose <u>means</u> taking the dual of a linear transformation.