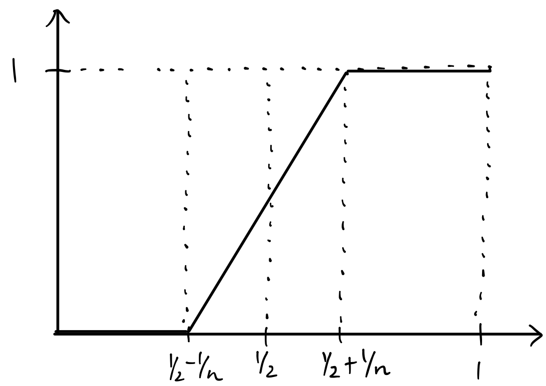
Tutorial 10 (rough notes)

Example L18-2 Consider $X = [0,1]$ and the sequence of $\overset{\text{continuous}}{\wedge}$ functions $f_n : X \longrightarrow \mathbb{R}$ given for $n \geq 4$ by



$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 - 1/n \\ n/2\,(x - 1/2 + 1/n) & 1/2 - 1/n < x < 1/2 + 1/n \\ 1 & 1/2 + 1/n \leq x \leq 1 \end{cases}$$

$(Ct(X,\mathbb{R}), d_p)$ $\int$

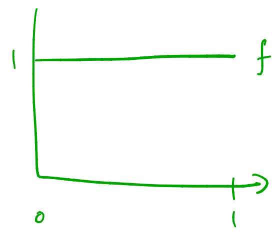
vs.

$(Cts(X,\mathbb{R}), d_\infty)$ sup.

The sequence $(f_n)_{n=4}^{\infty}$ converges <u>pointwise</u> to

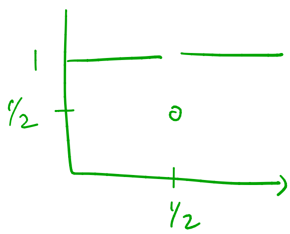
$$f(x) = \begin{cases} 1 & x > 1/2 \\ 1/2 & x = 1/2 \\ 0 & x < 1/2 \end{cases}$$

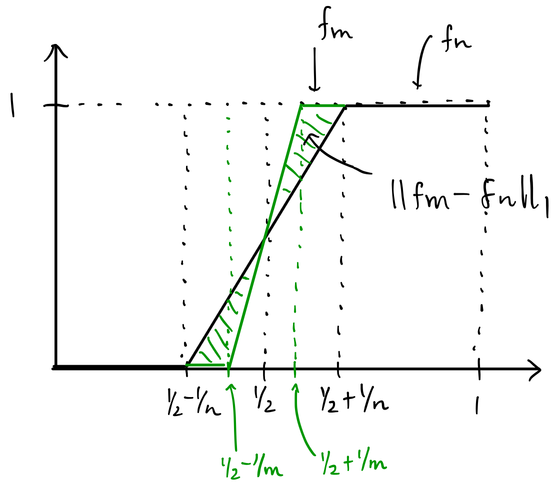
but this convergence is certainly not uniform (as the uniform limit of continuous functions is continuous). So $(f_n)_{n=4}^{\infty}$ <u>does not converge</u> in $(Cts(X,\mathbb{R}), d_\infty)$ (if it converged, it would have to be to f) and hence is not Cauchy (as this space is complete). However we claim the sequence <u>is</u> Cauchy in $(Ct(X,\mathbb{R}), d_p)$ but still does not converge, where throughout $1 \leq p < \infty$.



$$\int_a^b f\,dx = \int_a^b g\,dx$$

$$\int_a^b |f - g|\,dx = 0.$$





Observe that for $m \geq n \geq 4$ and $1 \leq p < \infty$, $|f_m - f_n| \leq |f - f_n|$ so

$$d_p(f_m, f_n) = \|f_m - f_n\|_p = \left\{ \int_X |f_m - f_n|^p \right\}^{1/p}$$

$$\leq \left\{ \int_X |f - f_n|^p \right\}^{1/p}$$

$$= \left\{ 2 \int_{1/2}^{1/2 + 1/n} |f - f_n|^p \right\}^{1/p}$$

$$= \left\{ 2 \int_{1/2}^{1/2 + 1/n} \left\{ 1 - n/2 \left(x - 1/2 + 1/n \right) \right\}^p \right\}^{1/p}$$

$u = 1 - n/2(x - 1/2 + 1/n)$

$$= 2^{1/p} \left\{ \frac{2}{n} \int_0^{1/2} u^p \, du \right\}^{1/p}$$

$$= 2^{2/p} \, n^{-1/p} \left\{ \underbrace{\int_0^{1/2} u^p \, du}_{>0} \right\}^{1/p}$$

Hence $(f_n)_{n=0}^{\infty}$ is Cauchy in $(Cts(X,\mathbb{R}), d_p)$. Now we claim this sequence does not converge in $(Cts(X,\mathbb{R}), d_p)$. To say $f_n \to g$ w.r.t. d_p says that $\forall \varepsilon > 0 \, \exists N \, \forall n \geq N \quad \int_X |f_n - g|^p < \varepsilon^p$.

<u>Problem</u> $\quad \int_X |f_n - g|^p$ small $\not\Rightarrow$ $f_n(x) - g(x)$ small any given x.

Suppose for a contradiction that $f_n \to g$ in $(Cts(X,\mathbb{R}), d_p)$. Then since the restriction function for $[c,d] \subseteq (\frac{1}{2}, 1]$ i.e. $(Cts(X,\mathbb{R}), d_p) \to (Cts([c,d],\mathbb{R}), d_p)$ is continuous so $f_n|_{[c,d]} \to g|_{[c,d]}$, i.e.

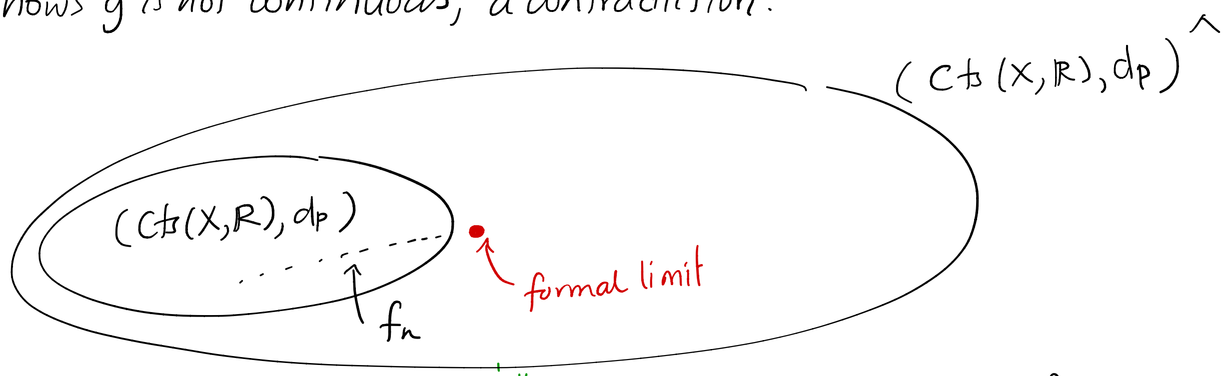
$$\forall \varepsilon > 0 \, \exists N \, \forall n \geq N \left(\left\{ \int_c^d |f_n - g|^p \right\}^{1/p} < \varepsilon \right)$$

norm on $(Cts([c,d],\mathbb{R}), \|-\|_p^{[c,d]})$

Claim that $\|1 - g\|_p^{[c,d]} < \varepsilon$ for any positive ε and hence $\|1 - g\|_p^{[c,d]} = 0$.
To see this let ε be given and find N such that $\|f_n - g\|_p^{[c,d]} < \varepsilon$ for $n \geq N$ and $f_n|_{[c,d]} = 1$ for $n \geq N$. Thus $\|1 - g\|_p^{[c,d]} < \varepsilon$, as required.
But since $\|-\|_p^{[c,d]}$ is a norm $1 - g = 0$ as claimed.

(ii) since $[c,d]$ was arbitrary $g(x) = 1$ for $x > \frac{1}{2}$.
(iii) similarly $g(x) = 0$ for $x < \frac{1}{2}$

This shows g is not continuous, a contradiction.



<u>Exercise L18-7</u> Give a counterexample to show that convergence $f_n \to f$ in $(Cts([a,b],\mathbb{R}), d_p)$ for $1 \leq p < \infty$ does <u>not</u> imply pointwise convergence. (Hint: $\frac{1}{2} - \frac{1}{m}$) (However $f_n \to f$ does imply pointwise convergence "almost everywhere" in a precise sense we will define later).