Tutorial 10 : Zorn's Lemma and vector spaces

The axiom of choice, in the form of Zorn's lemma, plays a foundational role in much of mathematics, and often in gruesomely technical ways. One important application is to show that for a field k, and vector space V over k, any linearly independent set $\mathcal{C} \subseteq V$ (possibly empty) is contained in a basis $\mathcal{B}$ of V (i.e. $\mathcal{C} \subseteq \mathcal{B}$). We say: any linearly independent set may be extended to a basis. In particular, taking $\mathcal{C} = \phi$, this shows that any vector space has a basis (which is not clear a priori unless V can be spanned by a finite set, in which case it is easy).

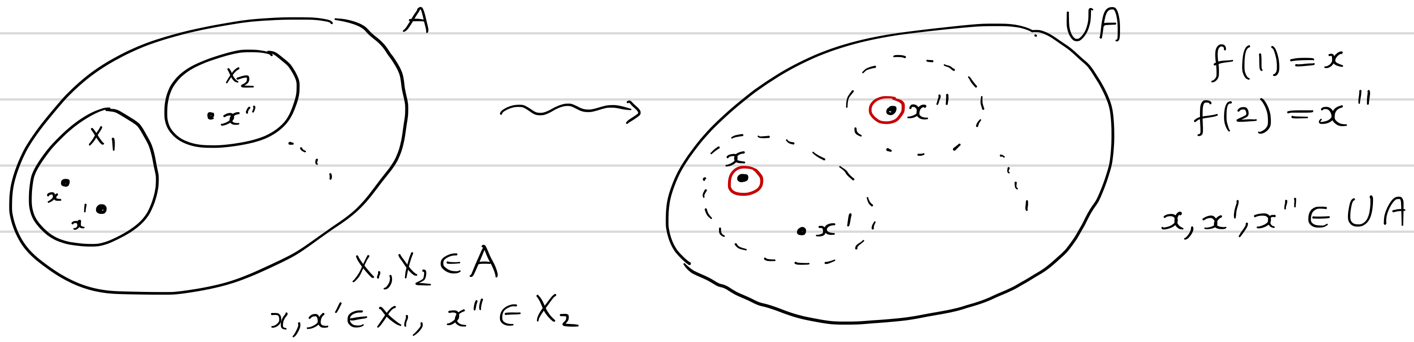
We begin with the statement of the axiom of choice. This does not require (or have!) a proof: that is why it is called an axiom. We assume it.

Axiom of Choice (AC) If $\{X_i\}_{i\in I}$ is an indexed family of sets and $X_i \neq \phi$ for each $i \in I$, then $\prod_{i\in I} X_i \neq \phi$.

Recall an indexed family $\{X_i\}_{i\in I}$ is just a surjective function $f: I \longrightarrow A$ with domain I, and $f(i) = X_i \in A$, and

$$\prod_{i\in I} X_i = \{ f : I \longrightarrow \cup A \mid f(i) \in X_i \text{ for all } i \in I\}.$$

Here $\cup A$ is the union set with $x \in \cup A \iff \exists a \in A\, (x \in a)$, or if you like, visualise $\cup A$ as obtained by "dissolving" the boundaries between every element of $\cup A$. A function $f \in \prod_{i\in I} X_i$ chooses one element per i:



<u>Def^n</u> Let $(P, \leq)$ be a partially ordered set. Then a subset $Q \subseteq P$ is called a <u>chain</u> if $\forall x, y \in Q \, (x \leq y \vee y \leq x)$. The empty set is vacuously a chain.

<u>Def^n</u> Let $(P, \leq)$ be a partially ordered set, $Q \subseteq P$. An element $x \in P$ is an <u>upper bound</u> for Q if $\forall y \in Q \, (y \leq x)$. An element $m \in P$ is <u>maximal</u> if $\forall y \in P \, (m \leq y \Rightarrow m = y)$.

<u>Example</u> In the powerset $P(\{0,1\})$ with inclusion, the subset $Q = \{\{0\}, \{1\}\}$ is <u>not</u> a chain.

<u>Zorn's Lemma</u> (ZL) If every chain in $(P, \leq)$ has an upper bound, then P contains a maximal element.

<u>Proof</u> By transfinite induction, which requires the language of ordinals and this does not fit in a tutorial. You can see the "Basic Set Theory" notes on my webpage for a self-contained presentation from scratch. Since Zorn's Lemma is <u>equivalent</u> (given the other axioms) to AC you are safe in taking the attitude that Zorn's Lemma is an axiom in your personal universe. $\square$

The statement of AC (which "obviously should be true") is included here in order to bully you into believing ZL (which looks less "obvious") via AC $\Leftrightarrow$ ZL.

<u>Theorem</u> Let k be a field and V a vector space over k. If $\mathcal{C} \subseteq V$ is linearly independent then there exists a basis $\mathcal{B}$ for V with $\mathcal{B} \supseteq \mathcal{C}$.

<u>Proof</u> Let P be the following set, partially ordered by inclusion

$$\mathcal{P} = \{ \mathcal{X} \subseteq V \mid \mathcal{X} \text{ is linearly independent and } \mathcal{X} \supseteq \mathcal{C} \}$$

Note that $\mathcal{C} \in \mathcal{P}$ so this set is nonempty ($\mathcal{C} = \phi$ is allowed). We claim every chain in $\mathcal{P}$ has an upper bound. Let $Q \subseteq \mathcal{P}$ be a chain. Then $\bigcup Q \subseteq V$ and we claim this is a linearly independent set. Given $\{v_1, \ldots, v_n\} \subseteq \bigcup Q$ we have $v_i \in \mathcal{X}_i$ for some $\mathcal{X}_i \in Q$, and by induction and the assumption that Q is a chain we have that one of the $\mathcal{X}_i$, say $\mathcal{X}_{i_0}$, contains all the others. But then $\{v_1, \ldots, v_n\} \subseteq \mathcal{X}_{i_0}$ must be LI. A set is LI iff. all its finite subsets are LI, so $\bigcup Q$ is LI. Since $\mathcal{X} \subseteq \bigcup Q$ whenever $\mathcal{X} \in Q$ this shows Q has an upper bound.

Hence by Zorn's Lemma $\mathcal{P}$ contains a maximal element Q^{max}. Suppose $\operatorname{span}(Q^{max}) \neq V$, say $w \in V \setminus \operatorname{span}(Q^{max})$. Then $\{w\} \cup Q^{max}$ is LI and hence $\{w\} \cup Q^{max} \in \mathcal{P}$, but since

$$\{w\} \cup Q^{max} \supsetneq Q^{max}$$

this contradicts maximality of Q^{max}. This contradiction shows Q^{max} is a basis for V, and by constuction it contains $\mathcal{C}$. □

<u>Corollary</u> Every vector space has a basis.

<u>Proof</u> $\mathcal{C} = \phi$. □

Addendum: (Secret uses of AC)

Let (X,d) be a metric space, $U \subseteq X$ an open subset, and for every $x \in U$ let $\varepsilon_x > 0$ be such that $B_{\varepsilon_x}(x) \subseteq U$.

You just used the axiom of choice! The assignment $x \mapsto \varepsilon_x$ is an element of

$$\prod_{x \in U} \{ \varepsilon \in \mathbb{R}_{>0} \mid B_\varepsilon(x) \subseteq U \}.$$

Each of the sets $\{\varepsilon \mid B_\varepsilon(x) \subseteq U\}$ is nonempty and you used that the product is nonempty, that is, you could simultaneously choose one ε for each x. This is the axiom of choice.

Suppose you could prove that for each x there was a unique $\varepsilon_x > 0$ satisfying $B_{\varepsilon_x}(x) \subseteq U$ and some other property $\phi(x, \varepsilon)$ (this stands for a formula in which x, ε occur).. The formula

$$\Phi(x, \varepsilon) \quad : \quad \varepsilon \in \mathbb{R}_{>0} \wedge B_\varepsilon(x) \subseteq U \wedge \phi(x, \varepsilon)$$

determines, by the Axiom of Comprehension, a subset

$$\{ (x, \varepsilon) \in U \times \mathbb{R}_{>0} \mid \Phi(x, \varepsilon) \text{ holds} \}$$

and this subset is precisely the graph of the function $U \longrightarrow \mathbb{R}_{>0}$ assigning to $x \in U$ this "special" ε_x. In this way we avoided invoking AC, but in practice this is not always possible (or convenient).