# Tutorial 11 : Constructing vectors in $L^2$

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We have defined $L^2(X,\mathbb{R})$ for an integral pair $(X, \int)$ to be the completion of the normed space $(Cts(X,\mathbb{R}), \|-\|_2)$. That means a vector $f \in L^2(X,\mathbb{R})$ is actually an equivalence class of Cauchy sequences

$$f = [(f_n)_{n=0}^{\infty}], \quad f_n : X \longrightarrow \mathbb{R} \text{ continuous.}$$

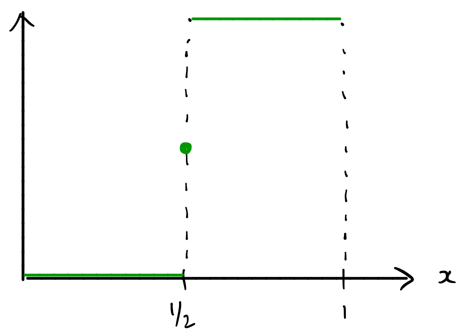
A more common way to introduce $L^2$-spaces is as equivalence classes of square-integrable (but not necessarily continuous!) functions on $X$. The purpose of today's tutorial is to reconcile these two definitions, so that when you learn about the Lebesgue integral you are prepared to reinterpret the contents of MAST30026 in that language.

FIRST: Example L18-2 from lectures. With $X = [0,1]$ this constructs a Cauchy sequence $(f_n)_{n=0}^{\infty}$ in $(Cts(X,\mathbb{R}), d_2)$ which does not converge, so that

$$f = [(f_n)_{n=0}^{\infty}] \in L^2(X,\mathbb{R})$$

is not in the image of the canonical inclusion $\iota : Cts(X,\mathbb{R}) \to L^2(X,\mathbb{R})$. Note that $f = \lim_{n\to\infty} \iota(f_n)$ in $L^2(X,\mathbb{R})$, and somehow this limit "wants" to be the function $f^{ptw} : [0,1] \longrightarrow \mathbb{R}$ which is the pointwise limit of the $f_n$'s :

$$f^{ptw}(x) = \begin{cases} 1 & x > 1/2 \\ 1/2 & x = 1/2 \\ 0 & x < 1/2 \end{cases}$$



However $f^{ptw}$ is not continuous so $f^{ptw} \notin \mathrm{Cts}(X,\mathbb{R})$. In a formal sense we will now explain $f$ "is" $f^{ptw}$, although there are subtleties. The following is adapted from L20. Set $X = [a,b]$ in the following

<u>Lemma L20-15</u> Suppose $g : [a,b] \longrightarrow \mathbb{R}$ is a function which is Riemann integrable on $[a,b]$. Then

$$\mathcal{O}_g : \mathrm{Cts}(X,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad \mathcal{O}_g(f) = \int_{[a,b]} fg$$

is continuous and linear.

<u>Proof</u> Linearity is a basic property of the integral. Continuity with respect to $\|-\|_2$ follows from the Hölder inequality (the proof of which goes through in the present case, with $g$ Riemann integrable but not necessarily continuous) since

$$\begin{aligned} \left| \int fg - \int f'g \right| &= \left| \int (f-f')g \right| \\ &\leq \int |(f-f')g| \\ &\leq \|f-f'\|_2 \|g\|_2. \quad \square \end{aligned}$$

$\ulcorner \|g\|_2 = \left\{ \int_{[a,b]} |g|^2 \right\}^{1/2} \lrcorner$

By the universal property of the completion of a normed space we know that any continuous linear $\mathcal{O}_g : \mathrm{Cts}(X,\mathbb{R}) \longrightarrow \mathbb{R}$ (with respect to $\|-\|_2$ on the domain) extends uniquely to a continuous linear $\Theta_g : L^2(X,\mathbb{R}) \longrightarrow \mathbb{R}$ (Theorem L18-9), as in:

$$\begin{tikzcd} L^2(X,\mathbb{R}) \arrow[r, dashed, "\Theta_g"] & \mathbb{R} \\ \mathrm{Cts}(X,\mathbb{R}) \arrow[u, hook, "\iota"] \arrow[ur, "\mathcal{O}_g"'] & \end{tikzcd}$$

By Theorem L20–13 there is a canonical continuous bilinear pairing

$$L^2(X,\mathbb{R}) \times L^2(X,\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\langle (a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \rangle = \lim_{n\to\infty} \int a_n b_n$$

so that for any $G \in L^2(X,\mathbb{R})$ there is a continuous linear map

$$\langle -, G \rangle : L^2(X,\mathbb{R}) \longrightarrow \mathbb{R}.$$

By the Riesz representation theorem (Corollary L20–14) <u>every continuous linear functional on $L^2(X,\mathbb{R})$ arises this way</u>, and in particular for any $g: X \longrightarrow \mathbb{R}$ which is Riemann integrable there exists a unique $\hat{g} \in L^2(X,\mathbb{R})$ such that

$$\langle -, \hat{g} \rangle = \Theta_g : L^2(X,\mathbb{R}) \longrightarrow \mathbb{R}$$

that is, for all continuous $h: X \longrightarrow \mathbb{R}$, supposing $\hat{g} = (g_n)_{n=0}^{\infty}$

$$\lim_{n\to\infty} \int h g_n = \Theta_g(h) = \int hg.$$

This vector $\hat{g}$ is the "avatar" of $g$ in $L^2$-space.

<u>Example</u> In the context of Example L18–2, the Riemann integrable $f^{ptw}$ induces $\Theta_{f^{ptw}} : L^2(X,\mathbb{R}) \longrightarrow \mathbb{R}$ which corresponds to the Cauchy sequence $f$, that is,

$$\widehat{f^{ptw}} = f$$

In conclusion, we have a diagram of injective linear maps $(X = [a,b])$

$$\begin{tikzcd}
 & L^2(X,\mathbb{R}) & \hat{g} \\
 Cts(X,\mathbb{R}) \arrow[ur, hook, "\iota"] \arrow[r, hook] & \{\text{Integrable } g : X \to \mathbb{R}\}/\text{a.e. equality} \arrow[u] & g \arrow[u, mapsto]
\end{tikzcd}$$

However not every element of $L^2(X,\mathbb{F})$ can be obtained as $\hat{g}$! The notion of Riemann integrability is artifically restrictive, the correct notion is <u>Lebesgue integrability</u>, and it is true that every vector in $L^2(X,\mathbb{F})$ represents a Lebesgue integrable function, so that there is an isomorphism

$$L^2(X,\mathbb{R}) \cong \frac{\{\text{Lebesgue square-integrable } g : X \to \mathbb{R}\}}{\text{almost everywhere equality.}}$$