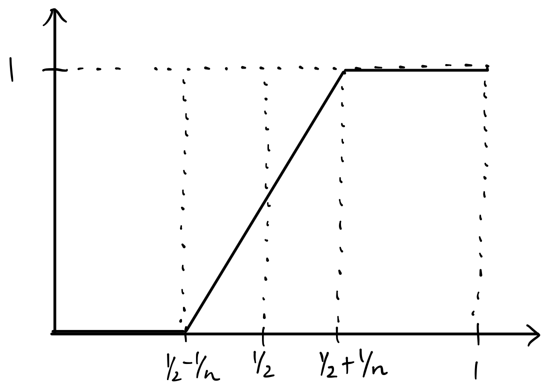
Tutorial 11

From Lecture 18 we know how to construct $L^2(X,\mathbb{R})$ for an integral pair $(X, \int_X)$ as a set of equivalence classes of Cauchy sequences $(f_n)_{n=0}^{\infty}$ where each $f_n : X \longrightarrow \mathbb{R}$ is continuous and the metric used to define Cauchy-ness is the one derived from the L^2-norm

$$\| f \|_2 = \left\{ \int_X |f|^2 \right\}^{1/2}$$

We know $L^2(X,\mathbb{R})$ is a Banach space and $(Cts(X,\mathbb{R}), \|-\|_2) \hookrightarrow (L^2(X,\mathbb{R}), \|-\|_2)$ as normed spaces, but L^2 space remains mysterious : <u>what are these vectors in $L^2(X,\mathbb{R}) \setminus Cts(X,\mathbb{R})$</u>? Are they just some formal objects or do they have some deeper significance? To examine this question we return to the topic of Tutorial 10.

<u>Example L18-2</u> Consider $X = [0,1]$ and the sequence of functions $f_n : X \longrightarrow \mathbb{R}$ given for $n \geq 4$ by



$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 - 1/n \\ n/2(x - 1/2 + 1/n) & 1/2 - 1/n < x < 1/2 + 1/n \\ 1 & 1/2 + 1/n \leq x \leq 1 \end{cases}$$

non-continuous but integrable

The sequence $(f_n)_{n=4}^{\infty}$ converges <u>pointwise</u> to $f(x) = \begin{cases} 1 & x > 1/2 \\ 1/2 & x = 1/2 \\ 0 & x < 1/2 \end{cases}$

$f \notin Cts(X,\mathbb{R})$

We proved $(f_n)_{n=4}^{\infty}$ is Cauchy in $(Cts(X,\mathbb{R}), d_2)$ but does not converge. Hence

$$\hat{f} := [(f_n)_{n=4}^{\infty}] \in L^2(X,\mathbb{R}) \setminus Cts(X,\mathbb{R})$$

$\exists g \in Cts(X,\mathbb{R})$

$(f_n)_{n=4}^{\infty} \sim (g)_{n=4}^{\infty}$

$d(f_n, g) \longrightarrow 0$

$f_n \rightarrow g.$

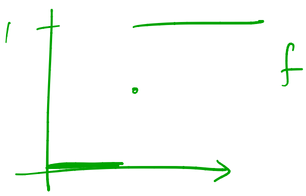
We will prove $L^2(X,\mathbb{R})$ is a Hilbert space $\langle , \rangle : L^2(X,\mathbb{R}) \times L^2(X,\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous in each variable and for $g, h \in Cts(X,\mathbb{R})$

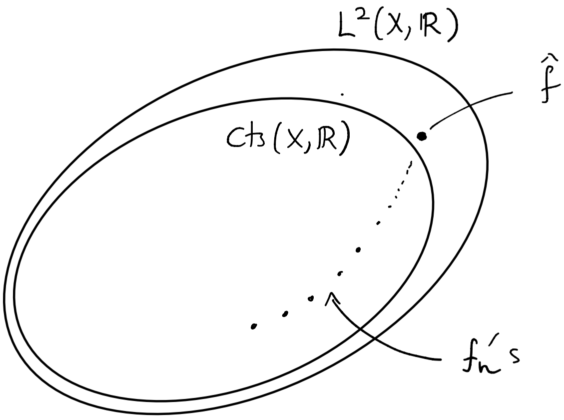
$$\langle g, h \rangle = \int_X gh.$$

Hence for $g \in Cts(X,\mathbb{R})$

$L^2(X,\mathbb{R}) \setminus Cts(X,\mathbb{R})$

$$\begin{aligned}
\langle g, \hat{f} \rangle &= \langle g, \lim_{n\to\infty} f_n \rangle \\
&= \lim_{n\to\infty} \langle g, f_n \rangle \\
&= \lim_{n\to\infty} \int_X g f_n \\
&= \lim_{n\to\infty} \left[\int_{1/2-1/n}^{1/2+1/n} g \left\{ \tfrac{n}{2}\left(x - \tfrac{1}{2} + \tfrac{1}{n}\right) \right\} + \int_{1/2+1/n}^{1} g \right] \\
&= \int_{1/2}^{1} g = \int_X gf
\end{aligned}$$





$\int_X gf$

$\langle g, f_n \rangle = \int_X g f_n$

$\langle g, f_1 \rangle = \int_X g f_1$

<u>Claim</u> We should think of $\hat{f}$ as "being" f. Moreover any Riemann integrable function can be represented in this way by a vector in $L^2(X,\mathbb{R})$.

There are several things to check before you should believe this

(1) Does $\hat{f}$ depend on the choice of approximation $(f_n)_{n=4}^{\infty}$ of f by continuous functions?

(2) How to find $(f_n)_{n=4}^{\infty}$ for a general Riemann integrable f?

(3) Is the function

$$\{\text{Riemann integrable } f\} \longrightarrow L^2(X, \mathbb{R})$$

injective? Surjective?

The key is:

Since this u is unique, we denote u by θ_η and call this the <u>representing element</u> for η.

Theorem L20-8 (Riesz representation theorem) Let $(H, \langle , \rangle)$ be a Hilbert space. If $\eta : H \longrightarrow \mathbb{F}$ is continuous and linear there exists a unique vector $u \in H$ with

$$\eta = \langle -, u \rangle.$$

Strategy
- Let $(X, \int_X) = ([0,1], \int_0^1)$.
- Let $f : X \longrightarrow \mathbb{R}$ be Riemann integrable
- Question: $\hat{f} \in L^2(X, \mathbb{R})$ "representing" f.
- $H = (L^2(X,\mathbb{R}), \langle , \rangle)$, $\eta : H \longrightarrow \mathbb{F}$ gives $\mathcal{O}_\eta \in L^2(X,\mathbb{R})$.

Lemma Given f integrable consider the function

$$\eta : Cts(X, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$\eta(g) = \int_0^1 gf \, dx.$$

is continuous and linear with respect to d_2, hence $\eta \in Cts(X,\mathbb{R})^\vee$.

Proof Linearity is clear. For continuity observe that

$$\Big| \underbrace{\int g f_{dx}}_{\eta(g)} - \underbrace{\int g' f_{dx}}_{\eta(g')} \Big| = \Big| \int (g - g') f_{dx} \Big|$$
$$\leq \int |g - g'| |f| \, dx \qquad \| (g-g') f \|_1$$
$$\text{Hölder inequality} \quad \leq \| g - g' \|_2 \, \| f \|_2$$
$$\{ \int_0^1 |f|^2 \}^{1/2}$$

Hence η is bounded $\| \eta \| \leq \| f \|_2$. $\square$

Let $\iota : Cts(X,\mathbb{R}) \longrightarrow L^2(X,\mathbb{R})$ be the inclusion. Since $\eta : Cts(X,\mathbb{R}) \to \mathbb{R}$ is continuous and linear there is a unique continuous linear η' making

$$\begin{tikzcd} L^2(X,\mathbb{R}) \arrow[r, dashed, "\eta'"] & \mathbb{R} \\ Cts(X,\mathbb{R}) \arrow[u, "\iota"] \arrow[ur, bend right, "\eta"'] & \end{tikzcd}$$

commute. Note

g_n continuous

$$\eta'(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} \eta(g_n) = \lim_{n\to\infty} \int_0^1 g_n f \, dx$$

By the Riesz representation theorem there is a unique representing element $\hat{f}$ for η'. That is, $\hat{f} \in L^2(X,\mathbb{R})$ and $\langle -, \hat{f} \rangle = \eta'$.
Equivalently

$\hat{f} = [(f_n)_{n=0}^{\infty}]$

$$\langle g, \hat{f} \rangle = \int_0^1 g f \, dx . \qquad \forall g \in Cts(X,\mathbb{R})$$

<u>Lemma</u> $\|\hat{f}\|_2 = \|f\|_2 = \left\{ \int_0^1 |f|^2 dx \right\}^{1/2}$.

$\| \; \lim_{n\to\infty} \|f_n\|_2$

$$\{\text{Riemann integrable}\} \xrightarrow{\Psi} L^2(X,\mathbb{R})$$

$$f \mapsto \hat{f}$$

- Ψ is not injective, $\mathrm{Ker}(\Psi) = \{$almost everywhere zero functions$\}$.
- Ψ is not surjective either!

$$\{\text{Lebesgue integrable}\} / \text{a.e. equality} \xrightarrow{\cong} L^2(X,\mathbb{R})$$