Tutorial 2 : Quotients and saturated sets

The aim of this tutorial is to make you comfortable with the quotient topology. Recall that if X is a topological space and $\sim$ is an equivalence relation then the quotient $X/\sim$ is the set of equivalence classes with the topology

$$\mathcal{T}_{X/\sim} = \{ U \subseteq X/\sim \mid p^{-1}(U) \text{ is open} \}$$

where $p : X \longrightarrow X/\sim$ is the quotient.

$\boxed{Q1}$ Suppose $f : X/\sim \longrightarrow Y$ is a function, Y another topological space. Prove f is continuous iff. $f \circ p$ is continuous.

Lemma There is a bijection between open subsets of $X/\sim$ and saturated open subsets of X, i.e. open subsets $U \subseteq X$ such that if $x \sim y$ and $x \in U$ then also $y \in U$.

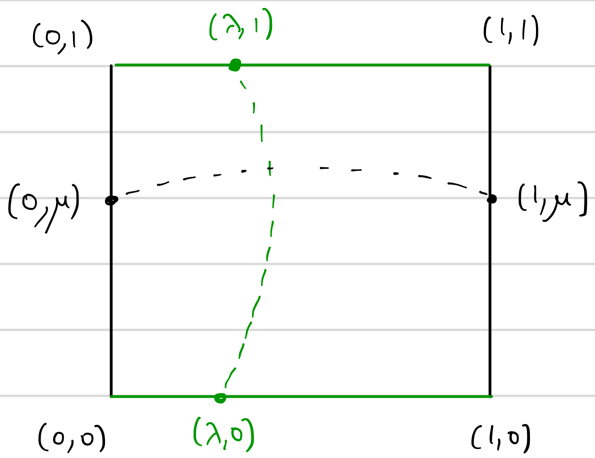
Proof (Proof A) The map $\mathcal{T}_{X/\sim} \longrightarrow \mathcal{T}_X$ sending $T \subseteq X/\sim$ to $p^{-1}(T)$ is injective since $T = p(p^{-1}(T))$, and clearly $p^{-1}(T)$ is saturated since if $x \sim y$ and $x \in p^{-1}(T)$ then $p(y) = p(x) \in T$ so $y \in p^{-1}(T)$. If $U \subseteq X$ is saturated we claim $U = p^{-1}(p(U))$. The inclusion $U \subseteq p^{-1}(p(U))$ is automatic, and for the reverse inclusion if $y \in p^{-1}(p(U))$ then $p(y) \in p(U)$, i.e. $p(y) = p(x)$ for some $x \in U$, thus $x \sim y$ and by saturatedness $y \in U$ as claimed. Hence the image of $\mathcal{T}_{X/\sim} \longrightarrow \mathcal{T}_X$ is precisely the set of saturated open sets.

(Proof B) Observe that by Lemma L6-2 the rows in the following commutative diagram are bijective (Σ is Sierpiński)

$$\begin{tikzcd}
\mathrm{Cts}(X/\sim, \Sigma) \arrow[r, "\cong"] \arrow[d, "(-)\circ p"'] & \mathcal{T}_{X/\sim} \arrow[d, "p^{-1}"] \\
\mathrm{Cts}(X, \Sigma) \arrow[r, dash, "\cong"] & \mathcal{T}_X
\end{tikzcd} \qquad (*)$$

So it suffices to prove the LHS vertical map is injective, and its image is the set of characteristic functions of saturated open sets. But p is surjective so injectivity is clear, and $\chi_U : X \longrightarrow \Sigma$ is in the image iff. $\chi_U(x) = \chi_U(y)$ whenever $x \sim y$ (by the universal property of the quotient), or equivalently $x \in U \Leftrightarrow y \in U$ whenever $x \sim y$, which is the definition of saturated. $\square$

For the rest of the tutorial we study the following example $X = [0,1]^2$ and $\sim$ the equivalence relation generated by the following pairs



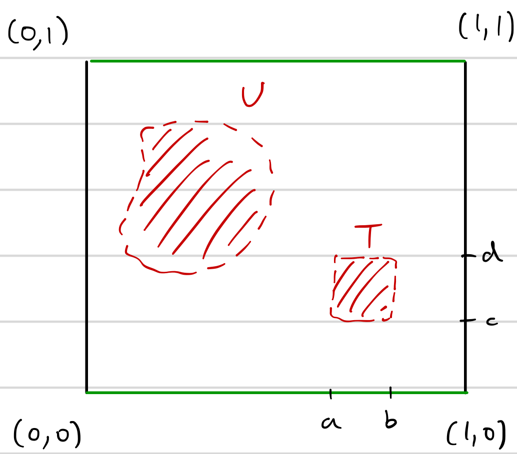
$(\lambda, 0) \sim (\lambda, 1) \qquad 0 \le \lambda \le 1$

$(0, \mu) \sim (1, \mu) \qquad 0 \le \mu \le 1$

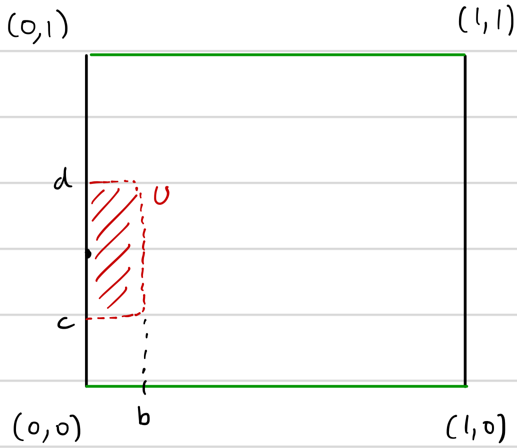
One checks that $\sim$ is the set

$$\begin{aligned}
&\{ ((\lambda,0),(\lambda,1)) \mid 0 \le \lambda \le 1 \} \\
\cup &\{ ((\lambda,1),(\lambda,0)) \mid 0 \le \lambda \le 1 \} \\
\cup &\{ ((0,\mu),(1,\mu)) \mid 0 \le \mu \le 1 \} \\
\cup &\{ ((1,\mu),(0,\mu)) \mid 0 \le \mu \le 1 \} \\
\cup &\{ ((\lambda,\mu),(\lambda,\mu)) \mid 0 \le \lambda, \mu \le 1 \} \\
\cup &\{ ((0,1),(1,0)), ((1,0),(0,1)), \\
&\quad ((0,0),(1,1)), ((1,1),(0,0)) \}
\end{aligned}$$

Let us study the saturated open subsets of X with respect to $\sim$.

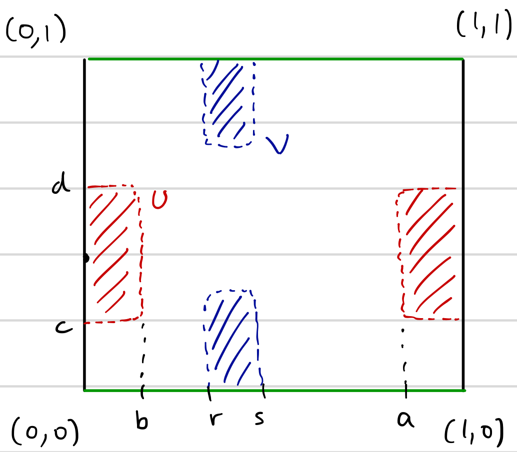


Any open set $U \subseteq [0,1]^2$ which does not meet the boundary is saturated. In particular $T = (a,b) \times (c,d)$ is saturated provided $0 < a < b < 1$, $0 < c < d < 1$.



$b \le 1$, $0 < c < d < 1$

The open set $[0,b) \times (c,d)$ is not saturated since $(0,x) \in U$ for $c < x < d$ but $(1,x) \notin U$.

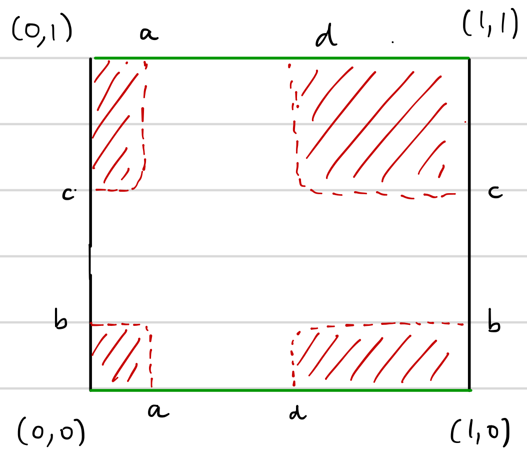


The open sets

$$U = [0,b) \times (c,d) \cup (a,1] \times (c,d)$$
$$V = (r,s) \times [0,t) \cup (r,s) \times (u,1]$$

are saturated $(0 < r < s < 1)$

What about the corners? An open square at any one, two or three corners cannot be open on its own.



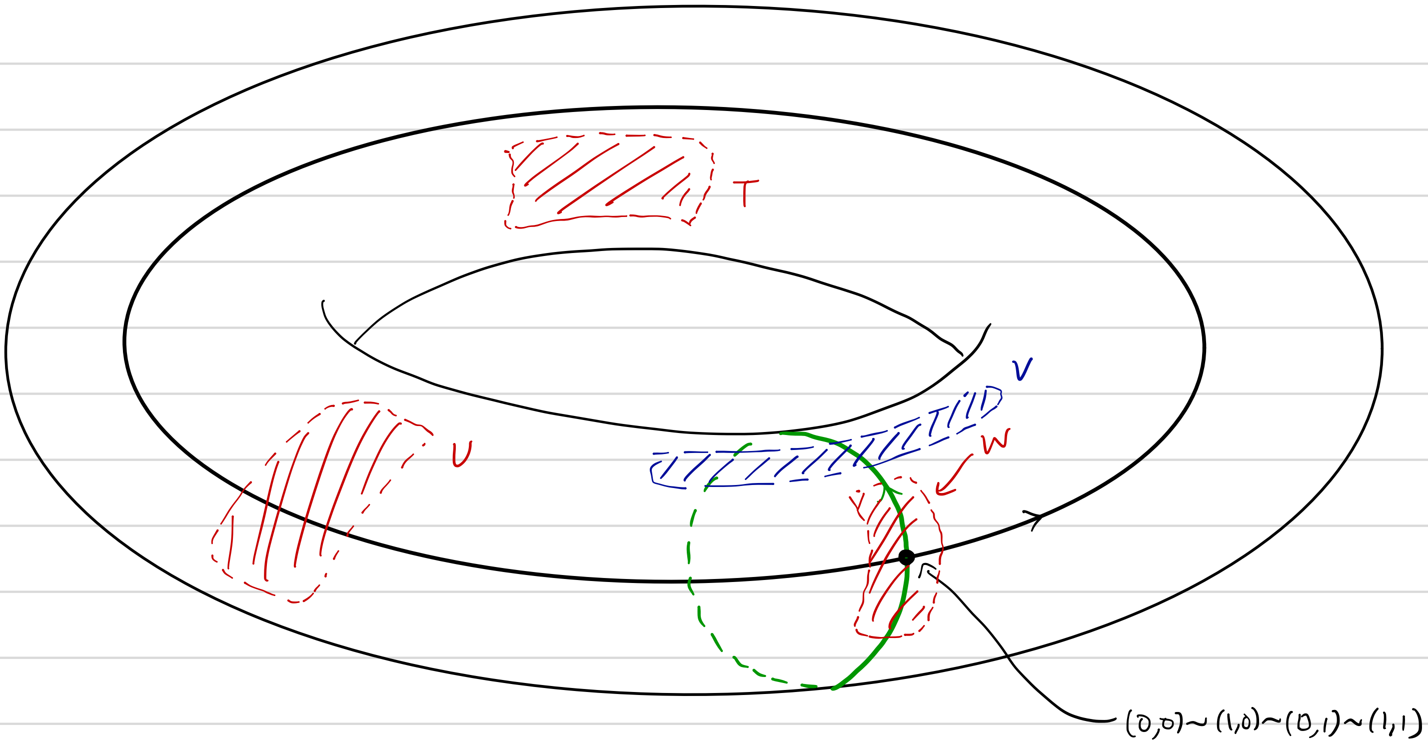
The open set

$$
\begin{aligned}
W = \ & [0,a) \times [0,b) \\
& \cup [0,a) \times (c,1] \\
& \cup (d,1] \times [0,b) \\
& \cup (d,1] \times (c,1]
\end{aligned}
$$

is saturated.

$\boxed{Q2}$ The open sets $T_{a,b,c,d}$, $U_{a,b,c,d}$, $V_{r,s,t,u}$ and $W_{a,b,c,d}$ form a basis for the set of saturated open sets, and thus induce a basis for the topology on $X/\sim$ (actually a smaller subset will do, can you see why?).

The quotient $X/\sim \; = [0,1]^2/\sim$ is of course (homeomorphic to) the <u>torus</u>.



$\boxed{Q3}$ Prove that $X/\sim \;\cong S^1 \times S^1$ where $S^1 \subseteq \mathbb{R}^2$ is the unit circle.

$\boxed{Sol^N}$ First we construct a bijective continuous map between these spaces. The idea is that you should always let universal properties do as much of the heavy lifting as you can: so we prefer to construct $X/\sim \;\longrightarrow S^1 \times S^1$ rather than $S^1 \times S^1 \longrightarrow X/\sim$.

Consider $f, g : [0,1]^2 \longrightarrow \mathbb{R}^2$ given by

$$f(x,y) = (\cos(2\pi x), \sin(2\pi x))$$
$$g(x,y) = (\cos(2\pi y), \sin(2\pi y))$$

both are continuous, and induce continuous maps $f, g : [0,1]^2 \longrightarrow S^1$ and hence by the universal property of the product, a continuous map $\tilde{\Phi} : [0,1]^2 \longrightarrow S^1 \times S^1$ defined by

$$\tilde{\Phi}(x,y) = (f(x,y), g(x,y)).$$

Since $\tilde{\Phi}$ identifies pairs related under $\sim$, we get by the universal property of the quotient a continuous map

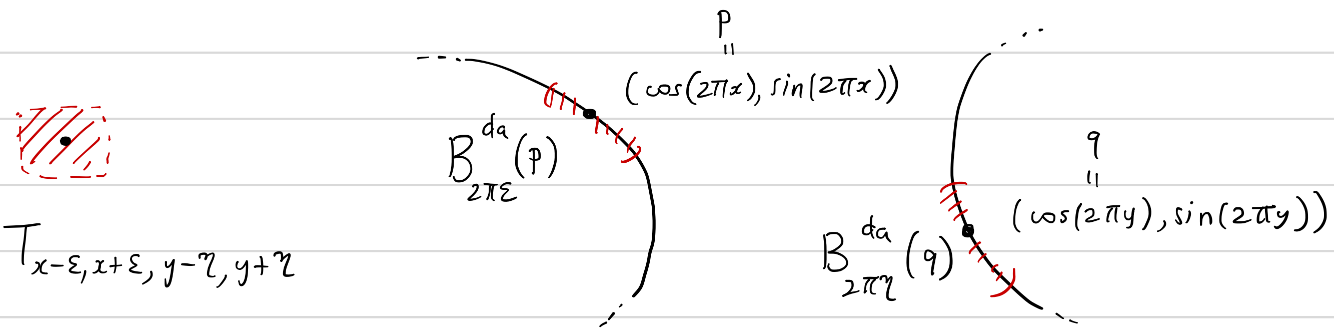
$$\Phi : [0,1]^2/\sim \;\longrightarrow S^1 \times S^1$$

It is easy to check it is bijective. Let the inverse function be Ψ, so we have only to show Ψ is continuous. But for this it suffices to show that the preimage under Ψ of any element of our chosen <u>basis</u> is open.

For this it is actually convenient to use the arc length metric d_a on S^1, with open balls $B_\varepsilon^{d_a}(\theta) \subseteq S^1$, since for example

$$\begin{aligned}\Psi^{-1}\left(T_{x-\varepsilon,x+\varepsilon,y-\eta,y+\eta}\right) &= \Phi\left(T_{x-\varepsilon,x+\varepsilon,y-\eta,y+\eta}\right)\\ &= \Phi\left(\overline{(x-\varepsilon,x+\varepsilon)\times(y-\eta,y+\eta)}\right)\\ &= B_\varepsilon^{d_a}(p)\times B_\eta^{d_a}(q)\subseteq S^1\times S^1\end{aligned}$$

which is open, as shown below:



The other cases are handled similarly. $\square$

$\boxed{Q4}$ (The Torus represents "bi-periodic" functions) Prove that for any space Y there is a bijection between continuous functions $f: \mathbb{R}\times\mathbb{R}\longrightarrow Y$ which are bi-periodic, in the sense that there exist $P_1, P_2 > 0$ with

$$f(x,y) = f(x+P_1, y) \qquad \forall x,y\in\mathbb{R}$$
$$f(x,y) = f(x, y+P_2) \qquad \forall x,y\in\mathbb{R}$$

and continuous functions from the torus $S^1\times S^1$ to Y.