Solutions

Q2 Let $\underline{v}_1, \ldots, \underline{v}_n$ be a basis of eigenvectors for A. If $\underline{v}_i, \underline{v}_j$ are eigenvectors for distinct eigenvalues $\lambda_i \neq \lambda_j$ then (assuming $\lambda_j \neq 0$ wlog)

$$\begin{aligned}
\underline{v}_i^T \underline{v}_j &= \underline{v}_i^T (\lambda_j^{-1} A \underline{v}_j) \\
&= \lambda_j^{-1} \underline{v}_i^T A \underline{v}_j \\
&= \lambda_j^{-1} (A \underline{v}_i)^T \underline{v}_j \\
&= \lambda_j^{-1} \lambda_i \underline{v}_i^T \underline{v}_j
\end{aligned}$$

Hence $(1 - \lambda_j^{-1} \lambda_i) \underline{v}_i^T \underline{v}_j = 0$ and since $\lambda_i \neq \lambda_j$ we may conclude that $\underline{v}_i^T \underline{v}_j = 0$. For each eigenvalue λ we apply Gram-Schmidt to produce an orthonormal basis of eigenvectors for the space of eigenvectors. Let Q be the matrix whose columns are the eigenvectors produced by this process (for all λ). Then $Q^T Q = I_n$ by construction, and $AQ = DQ$ for D diagonal, so $Q^T A Q = D$.

$\boxed{Q6}$ We have $(\mathbb{R}^n, B_P) \sim (\mathbb{R}^n, B_{P'})$

$$\iff \exists \text{ invertible } Q \text{ with } B_{P'}(Q\underline{v}, Q\underline{w}) = B_P(\underline{v}, \underline{w}) \quad \forall \underline{v}, \underline{w}$$

$$\iff \exists \text{ invertible } Q \text{ with } (Q\underline{v})^T P' Q \underline{w} = \underline{v}^T P \underline{w} \quad \forall \underline{v}, \underline{w}.$$

$$\iff \exists \text{ invertible } Q \text{ with } \underline{v}^T Q^T P' Q \underline{w} = \underline{v}^T P \underline{w} \quad \forall \underline{v}, \underline{w}$$

$$\iff \exists \text{ invertible } Q \text{ with } Q^T P' Q = P.$$

Note Q need not be orthogonal! i.e. maybe $Q^T \neq Q^{-1}$. So this does not necessarily say P', P are related by a change of basis.

$\boxed{Q7}$ We have $(\mathbb{R}^n, B_P)$ for some invertible symmetric P. Let Q_1 be orthogonal s.t. $Q_1^T P Q_1$ is $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Then by Q6

$$(\mathbb{R}^n, B_P) \sim (\mathbb{R}^n, B_D).$$

Choosing the order of the columns of Q correctly, we may assume that for some $a, b \geq 0$ with $a+b=n$ we have $\lambda_1, \dots, \lambda_a > 0$ and $\lambda_{a+1}, \dots, \lambda_n < 0$. Then with $Q_2 = \text{diag}(|\lambda_1|^{-1/2}, \dots, |\lambda_n|^{-1/2})$ we have

$$Q_2^T D Q_2 = \text{diag}(\overbrace{1, \dots, 1}^{a}, \overbrace{-1, \dots, -1}^{b})$$

and so $(\mathbb{R}^n, B_D) \sim (\mathbb{R}^n, B^{a,b})$ as claimed.

For uniqueness, suppose $(\mathbb{R}^n, B^{a,b}) \sim (\mathbb{R}^n, B^{a',b'})$ with $a > a'$. Then we have an invertible Q s.t.

$$B^{a',b'}(Q\underline{x}, Q\underline{x}) = B^{a,b}(\underline{x}, \underline{x}) \qquad \forall \underline{x} \in \mathbb{R}^n$$

$$(*)$$

$$w_1^2 + \cdots + w_{a'}^2 - w_{a'+1}^2 - \cdots - w_n^2$$
$$= x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_n^2 \qquad \forall \underline{x} \in \mathbb{R}^n$$

where $\underline{w} = Q\underline{x}$, so each w_i is a linear function of the x_j. Let us fix $x_{a+1} = \cdots = x_n = 0$ and call the corresponding subspace $Y \subseteq \mathbb{R}^n$. Consider the linear map

$$\begin{array}{ccccccc} Y & \xrightarrow{\text{inc}} & \mathbb{R}^n & \xrightarrow{Q} & \mathbb{R}^n & \xrightarrow{\text{projection}} & \mathbb{R}^{a'} \\ \underline{x} & \longmapsto & (\underline{x}, \underline{0}) & \longmapsto & \underline{w} & \longmapsto & (w_1, \dots, w_{a'}) \end{array}$$

Now $\dim(Y) = a > a' = \dim(\mathbb{R}^{a'})$ so this linear transformation must have a nonzero kernel. Let $(y_1, \dots, y_a)$ be nonzero and in the kernel. Evaluating $(*)$ on this point yields

$$y_1^2 + \cdots + y_a^2 = -w_{a'+1}(\underline{y})^2 - \cdots - w_n(\underline{y})^2$$

which is a contradiction since the LHS is strictly positive and the RHS is ≤ 0. This contradiction shows $a = a'$ as claimed.