MAST30026 Tutorial 2

Q1 What does the Gram-Schmidt process do? (Do not give the algorithm, just describe at a high level what the input and output of the algorithm are).

Q2 Any real symmetric matrix A is diagonalisable (you may assume this). Show that there is an <u>orthogonal</u> matrix Q with $Q^T A Q$ diagonal.

<u>Definition</u> If V is a vector space, a function $B : V \times V \longrightarrow \mathbb{R}$ is <u>bilinear</u> if

$$\begin{array}{lll} \text{(i)} & B(x+x', y) = B(x,y) + B(x',y) & \forall x, x', y \in V \\ \text{(ii)} & B(x, y+y') = B(x,y) + B(x,y') & \forall x, y, y' \in V \\ \text{(iii)} & B(\lambda x, y) = B(x, \lambda y) = \lambda B(x,y) & \forall x, y \in V \ \ \forall \lambda \in \mathbb{R}. \end{array}$$

We say B is <u>symmetric</u> if $B(x,y) = B(y,x)$ for all $x, y \in V$.

Q3 Prove that there is a bijection between bilinear forms on V and linear maps $V \longrightarrow V^*$ where V^* denotes the dual space, where the linear map f_B associated to B is $f_B(x)(y) = B(x,y)$.

Q4 Suppose V is finite-dimensional. A symmetric bilinear form B on V is called <u>nondegenerate</u> if f_B is an isomorphism $V \xrightarrow{\cong} V^*$. Write down the definition of nondegeneracy purely in terms of pairs (x,y) for which $B(x,y) = 0$.

Q5 Prove that if B is a symmetric bilinear form on $\mathbb{R}^n$ there is a unique symmetric $P \in M_n(\mathbb{R})$ with $B(\underline{v}, \underline{w}) = \underline{v}^T P \underline{w}$ for all $\underline{v}, \underline{w} \in \mathbb{R}^n$.
We denote this bilinear form $(\underline{v}, \underline{w}) \longmapsto \underline{v}^T P \underline{w}$ by B_P.
What condition does nondegeneracy of B_P place on P?

A *quadratic space* is a pair (V, B) where V is a finite-dimensional vector space and B is a symmetric bilinear form on V. Two quadratic spaces $(V,B),(V',B')$ are *equivalent* and we write $(V,B)\sim(V',B')$ if there is a linear isomorphism $F: V \to V'$ with $B'(Fx,Fy) = B(x,y)$ for all $x,y \in V$.

$\boxed{Q6}$ Given symmetric matrices $P, P' \in M_n(\mathbb{R})$ write the condition

$$(\mathbb{R}^n, B_P) \sim (\mathbb{R}^n, B_{P'})$$

purely in terms of matrices.

$\dim V = n$

$\boxed{Q7}$ (Sylvester's law of inertia) Prove that any quadratic space (V, B) with B nondegenerate is equivalent, for a unique pair of integers $a,b \geq 0$ with $a+b=n$, to the pair $(\mathbb{R}^n, B^{a,b})$ where $B^{a,b}$ is the bilinear form associated to the matrix

$$\mathrm{diag}(\underbrace{+1,+1,\dots,+1}_{a},\underbrace{-1,\cdots,-1}_{b}).$$

The pair (a,b) is called the *signature* of B, and the group $O(a,b)$ is the associated isotropy group

$$O(a,b) := \{ F: V \to V \mid F \text{ is a linear iso., and } B(Fx,Fy) = B(x,y)\ \forall x,y \in V \}.$$

Remark Minkowski space has signature $(3,1)$, hence the *Lorentz group* is denoted $O(3,1)$ (or in lectures $O(2,1)$ since we had two space directions).