Tutorial 4 : The circle as a group

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This tutorial is a (re) examination of the circle, as a group. This will be important when we arrive at the Hilbert space $L^2(S^1)$ of functions on S^1.

Defn A topological group is a set X equipped both as a topological space $(X, \mathcal{J})$ and as a group $(X, \cdot, e)$ such that

(i) The function $\cdot : X \times X \longrightarrow X$ is continuous.

(ii) The function $(-)^{-1} : X \longrightarrow X$ is continuous.

An isomorphism of topological groups is a homeomorphism that is also an isomorphism of groups.

Q1 Prove that $(\mathbb{R}, +, 0)$ is a topological group

Q2 Prove that $(GL(2,\mathbb{R}), \cdot, I_2)$ is a topological group, where $GL(2,\mathbb{R}) \subseteq M_2(\mathbb{R})$ is the set of invertible matrices. Hence show $SO(2,\mathbb{R})$ is also a topological group. (what about the general case?)

We write $GL(2)$, $SO(2)$ for $GL(2,\mathbb{R})$, $SO(2,\mathbb{R})$.

Q3 Prove that the bijection $S^1 \longrightarrow SO(2)$ sending $(\cos\theta, \sin\theta)$ to the rotation matrix R_θ is a homeomorphism, where $S^1 \subseteq \mathbb{R}^2$ is the unit circle. Hence S^1 is a topological group with structure maps

$$S^1 \times S^1 \cong SO(2) \times SO(2) \xrightarrow{\ \cdot\ } SO(2) \cong S^1$$

$$S^1 \cong SO(2) \xrightarrow{(-)^{-1}} SO(2) \cong S^1$$

Q4 Prove S^1 is isomorphic as a topological group to

$$U(1) = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \subseteq \mathbb{C}$$

under multiplication, with the induced topology.

Q5 Let G be a topological group and $H \subseteq G$ a normal subgroup. Let $\sim$ be the equivalence relation $g \sim g'$ iff. $\exists\, h \in H$ with $g = hg'$, so that $G/H := G/\sim$. Prove that G/H is a topological group when given the quotient topology.

Q6 Prove $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.

Note that S^1 is not "canonically" a group, since nobody tells you where to start measuring your angles from, so it is more accurate to say $SO(2)$, $U(1)$, $\mathbb{R}/\mathbb{Z}$ are topological groups, all homeomorphic to S^1 as spaces.