Tutorial 4 : One-point compactification

The two "nicest" classes of spaces are metrisable spaces, and compact Hausdorff spaces. After that, the next best kind of space are subspaces of these kinds of spaces: subspaces of metrisable spaces are metrisable but subspaces of compact spaces need not be compact, so in this latter case we obtain a new and interesting class of spaces: the locally compact Hausdorff spaces. In the second half of this course we will prove many theorems about this class of spaces. In this tutorial we develop the theory of locally compact spaces, as an extended exercise in the basic properties of compactness itself.

Def^n A topological space X is locally compact if for every $x \in X$ there exists an open set U and compact set K with $x \in U \subseteq K$.

Clearly any compact space is locally compact.

Example (i) $\mathbb{R}^n$ is locally compact (but not compact) since any $x \in \mathbb{R}^n$ lies in some $(a_1, b_1) \times \cdots \times (a_n, b_n)$ which is contained in the compact set $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

(ii) $\mathbb{Q}$ is not locally compact (so subspaces of locally compact spaces need not be locally compact).

Def^n A topological space X is Hausdorff if for any pair $x, y \in X$ of distinct points there exist open $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (developed in L11)

Remark (i) In a Hausdorff space points are closed.

(ii) Any subspace of a Hausdorff space is Hausdorff.

(iii) A compact subspace of a Hausdorff space is closed.

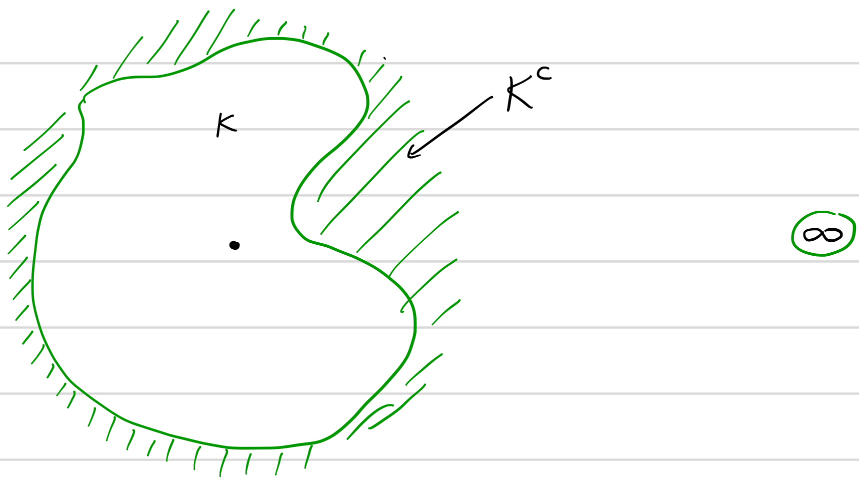
$\boxed{Q1}$ Any closed subspace of a locally compact space is locally compact.

<u>Theorem</u> If X is locally compact Hausdorff then it is homeomorphic to a subspace of a compact Hausdorff space. (e.g. $X = \mathbb{R}$)

We define for locally compact Hausdorff X a space $\tilde{X}$ that we call the <u>one-point compactification</u>, which is compact Hausdorff and has X as a subspace.

<u>Def^n</u> As a set $\tilde{X} = X \amalg \{\infty\}$. A subset $U \subseteq \tilde{X}$ which does not contain ∞ is open iff. it is open in X, and $U \subseteq \tilde{X}$ which contains ∞ is open iff. there is $K \subseteq X$ compact with $U = K^c \amalg \{\infty\}$.

<u>Example</u> $X = \mathbb{R}^2$ is locally compact



The set $K^c \amalg \{\infty\}$ is an open neighborhood of ∞ in $\tilde{X}$. By Heine-Borel a compact subset $K \subseteq \mathbb{R}^2$ must be closed and bounded, so this picture is actually generic, i.e. it describes all open neighborhoods of ∞.

$\boxed{Q2}$ Prove this is a topology on $\widetilde{X}$.

$\boxed{Q3}$ Prove $\widetilde{X}$ is compact and Hausdorff.

$\boxed{Q4}$ Prove the canonical map $X \longrightarrow \widetilde{X}$ induces a homeomorphism between X and a subspace of $\widetilde{X}$ (hence proving the theorem).

This space $\widetilde{X}$ is called the <u>one-point compactification</u> of X. It is not a very smart construction, the Stone-Čech compactification is better, but the one-point compactification has its uses.

$\boxed{Q4}$ Give a characterisation of the set of continuous functions $f: X \longrightarrow \mathbb{R}$ which extend to $\widetilde{X}$, that is, for which $F: \widetilde{X} \longrightarrow \mathbb{R}$ continuous exists making the diagram below commute:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, dashed, "F"] \\ & \mathbb{R} \end{tikzcd}$$

This class of functions will be necessary for defining the Hilbert spaces $L^2(\mathbb{R}^d)$ of square-integrable functions on Euclidean space.

$\boxed{\text{Solution to Q2}}$ The empty set does not contain ∞ and is open in X so $\phi \in \mathcal{T}_{\tilde{X}}$. The set $\tilde{X}$ contains ∞ but $\tilde{X} = \phi^c \amalg \{\infty\}$ and ϕ is compact (told you we would have to care about this). Suppose $\{U_i\}_{i\in I}$ is an indexed family of open sets in $\tilde{X}$, then let $J = \{ i \in I \mid \infty \in U_i \}$ and write $(J^c = I \setminus J)$

$$\bigcup_{i\in I} U_i = \bigcup_{j\in J} U_j \cup \bigcup_{j\in J^c} U_j$$

For $j \in J$ let $K_j \subseteq X$ be compact with $U_j = K_j^c \amalg \{\infty\}$. Then, observing that U_j is open in X for $j \in J^c$, (we may assume $J \neq \phi$, otherwise it is clear)

$$\begin{aligned}\bigcup_{j\in J} U_j \cup \bigcup_{j\in J^c} U_j &= \bigcup_{j\in J} K_j^c \cup \{\infty\} \cup \bigcup_{j\in J^c} U_j \\ &= \Big(\bigcup_{j\in J} K_j^c \cup \bigcup_{j\in J^c} U_j\Big) \cup \{\infty\} \\ &= \Big(\bigcap_{j\in J} K_j \cap \bigcap_{j\in J^c} U_j^c\Big)^c \cup \{\infty\}.\end{aligned}$$

But each $K_j \subseteq X$ is compact hence closed, so $\bigcap_{j\in J} K_j \cap \bigcap_{j\in J^c} U_j^c$ is a closed subspace of a compact space K_{j_0} (pick $j_0 \in J$, and note that if $J = \phi$ there is nothing to prove) hence compact, so we are done: $\bigcup_{j\in J} U_j$ is open.

If $U, V \subseteq \tilde{X}$ are open there are four cases:

(i) $U, V \subseteq X$ are open, then $U \cap V$ is open.

(ii) $U = K^c \amalg \{\infty\}$, $V \subseteq X$ is open, $U \cap V = K^c \cap V \subseteq X$ is open.

(iii) $U \subseteq X$, $V = K^c \amalg \{\infty\}$ same as (ii).

(iv) $U = K^c \amalg \{\infty\}$, $V = L^c \amalg \{\infty\}$ with K, L compact, then $U \cap V = (K \cup L)^c \amalg \{\infty\}$ with $K \cup L$ compact.

which shows $U \cap V$ is open in $\tilde{X}$, hence $\mathcal{T}_{\tilde{X}}$ is a topology. $\square$

$\boxed{\text{Solution to Q3}}$ Let $\{U_i\}_{i\in I}$ be an open cover of $\tilde{X}$, and suppose $\infty \in U_{i_0}$. Then $U_{i_0} = K^c \amalg \{\infty\}$ for some compact $K \subseteq X$. The open sets $\{U_i\}_{i\in I}$ cover K, so let $J \subseteq I$ be finite with $K \subseteq \bigcup_{j\in J} U_j$. Then we have

$$\begin{aligned}\tilde{X} &= X \cup \{\infty\} \\ &= K \cup K^c \cup \{\infty\} \\ &= K \cup U_{i_0} \\ &\subseteq \left(\bigcup_{j\in J} U_j\right) \cup U_{i_0}\end{aligned}$$

so $J \cup \{i_0\}$ is a finite subcover. To prove $\tilde{X}$ is Hausdorff, it suffices to consider $x \in X$, and observe that since X is locally compact there exists $x \in U \subseteq K$ with U open, K compact, but then $x \in U$ and $\infty \in K^c \amalg \{\infty\}$ are disjoint open neighbourhoods of x, ∞ in $\tilde{X}$. $\square$

$\boxed{\text{Solution to Q4}}$ The map $X \xrightarrow{f} \tilde{X}$ is clearly continuous, since if $K \subseteq X$ is compact then K is closed hence K^c is open. It is a bijection onto its image, and a homeomorphism since if $U \subseteq X$ is open then $f(U)$ is open by definition. $\square$

$\boxed{\text{Solution to Q4}}$ Let $f : X \longrightarrow \mathbb{R}$ continuous be given. The data of F is just some real number $\lambda := F(\infty)$, and the fact that F is continuous is equivalent to the constraint that for all $\varepsilon > 0$ the set

$$F^{-1} B_\varepsilon(\lambda) \subseteq \tilde{X}$$

is open. But this is equivalent to compactness of the set $\{x \in X \mid |f(x) - \lambda| \geqslant \varepsilon\}$. So we have our characterisation:

a continuous function $f: X \longrightarrow \mathbb{R}$ extends to $\tilde{X}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $f - \lambda$ vanishes at infinity, by which we mean that for all $\varepsilon > 0$ the set $\{x \in X \mid |f(x) - \lambda| \geqslant \varepsilon\}$ is compact.