Tutorial #4

This tutorial is a (re) examination of the circle, as a group. This will be important when we arrive at the Hilbert space $L^2(S^1)$ of functions on S^1.

Def^n A topological group is a set X equipped both as a topological space $(X, \mathcal{T})$ and as a group $(X, \bullet, e)$ such that

(i) The function $\bullet : X \times X \longrightarrow X$ is continuous.

(ii) The function $(-)^{-1} : X \longrightarrow X$ is continuous.

An isomorphism of topological groups is a homeomorphism that is also an isomorphism of groups.

$\boxed{Q1}$ Prove that $(\mathbb{R}, +, 0)$ is a topological group

$\boxed{Q2}$ Using $S^1 := [0,1]/\sim$ where $0 \sim 1$ give S^1 the structure of a topological group, isomorphic to the topological group

$$U(1) = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \subseteq \mathbb{C}$$

under multiplication.

$\boxed{Q3}$ Let G be a topological group and $H \subseteq G$ a normal subgroup. Let $\sim$ be the equivalence relation $g \sim g'$ iff. $\exists\, h \in H$ with $g = hg'$, so that $G/H := G/\sim$. Prove that G/H is a topological group when given the quotient topology.

$\boxed{Q4}$ Prove $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.