Tutorial 5 : Spaces of paths

10/9/19
updated 9/9/20

In the lectures we have started to develop the theory of function spaces and in this tutorial we aim to add to our intuition about these spaces with an extended study of the example of the space of paths in a topological space Y.

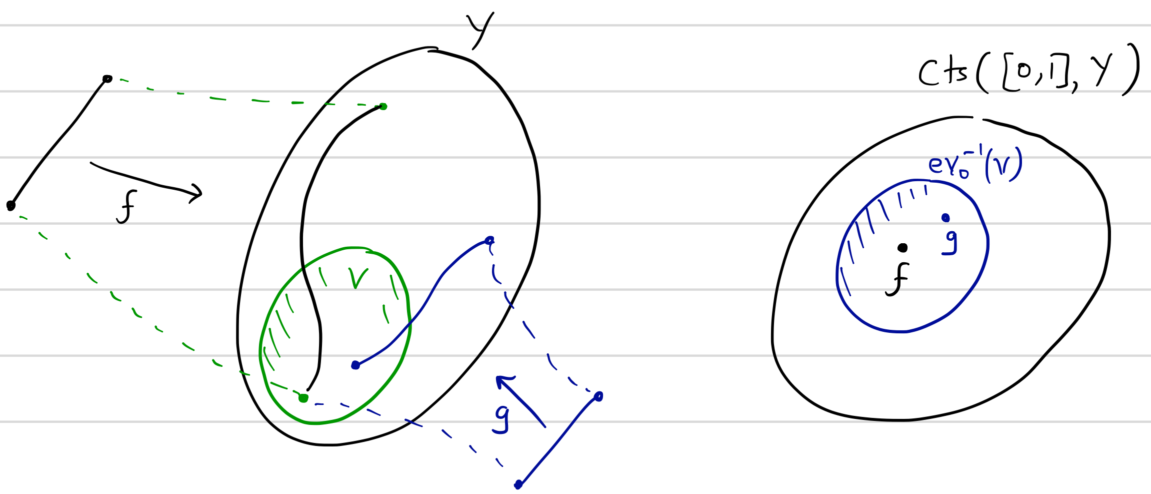
A path in Y is a continuous function $f : [0,1] \longrightarrow Y$. Such paths form a set $Cts([0,1], Y)$. For the purposes of this tutorial we take as given that there is a topology on this space (the compact-open topology) such that for all $x \in [0,1]$ the evaluation function

$$ev_x : Cts([0,1], Y) \longrightarrow Y$$
$$ev_x(f) = f(x)$$

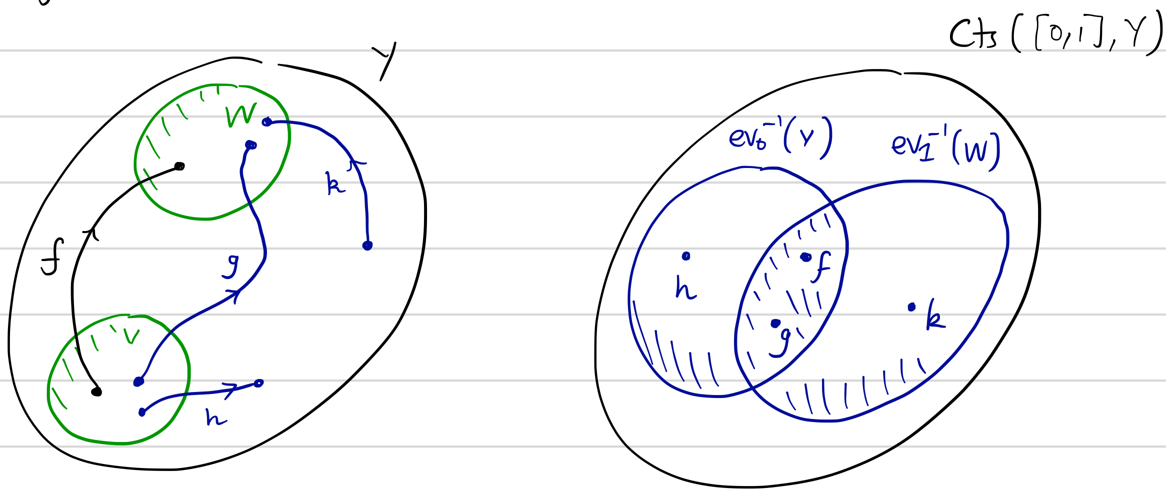
is continuous. Thus paths in Y are the points of the space $Cts([0,1], Y)$. What are the open sets? Well, in particular for $V \subseteq Y$ open the subset

$$ev_0^{-1}(V) = \{ f : [0,1] \longrightarrow Y \text{ cts} \mid f(0) \in V \}$$

is open in the space of paths. So any particular path starting in V has an open neighborhood consisting of all the paths starting in V.



We can form smaller open neighborhoods of f by imposing more constraints, e.g. if $W \subseteq Y$ is open and $f(1) \in W$ then $ev_0^{-1}(V) \cap ev_1^{-1}(W)$ is a smaller neighborhood of f in $Cts([0,1], Y)$.

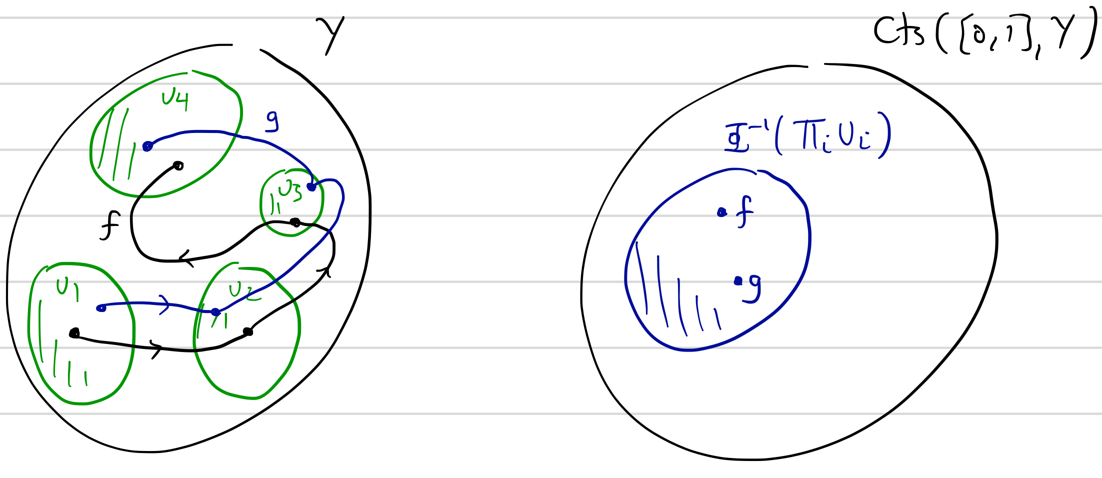


More generally, if $0 \le a_1 < a_2 < \cdots < a_n \le 1$ then the continuous functions $\{ev_{a_i} : Cts([0,1], Y) \longrightarrow Y\}_{i=1}^{n}$ induce a continuous map

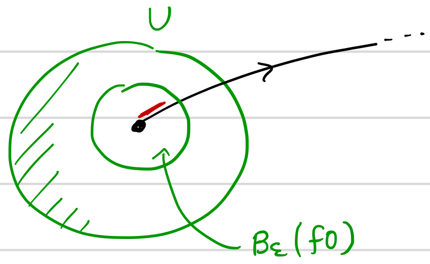
$$\Phi : Cts([0,1], Y) \longrightarrow \prod_{i=1}^{n} Y$$

$$\Phi(f) = (f(a_1), \ldots, f(a_n))$$

and for open sets $U_1, \ldots, U_n \subseteq Y$ the preimage $\Phi^{-1}(\prod_{i=1}^{n} U_i)$ is open.

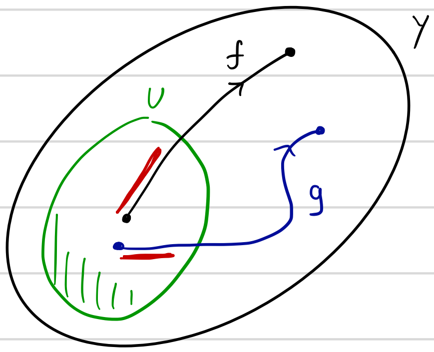


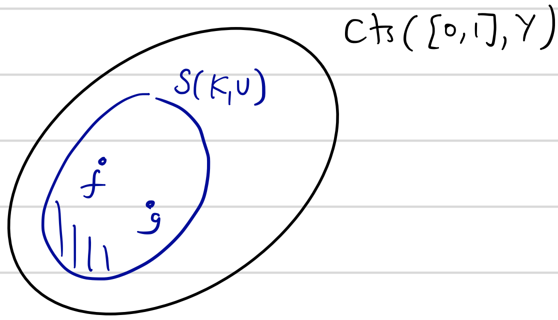
Now observe that if a path f starts in an open set $U \subseteq Y$ and Y is a metric space, there is some closed interval $K = [0, r]$ such that $f(K) \subseteq U$ (in words: if f starts in U there is some initial segment of f inside U. The point of this exercise is to develop intuition, so do not get hung up on the assumption Y is a metric space)



This seems like a reasonable way of prescribing paths "similar to f" which generalises the above. Rather than just impose $g(0) \in U$ (i.e. $g(K) \subseteq U$ for K a point) we can impose $g(K) \subseteq U$ Let us write (with $K = [0, r]$ as above)

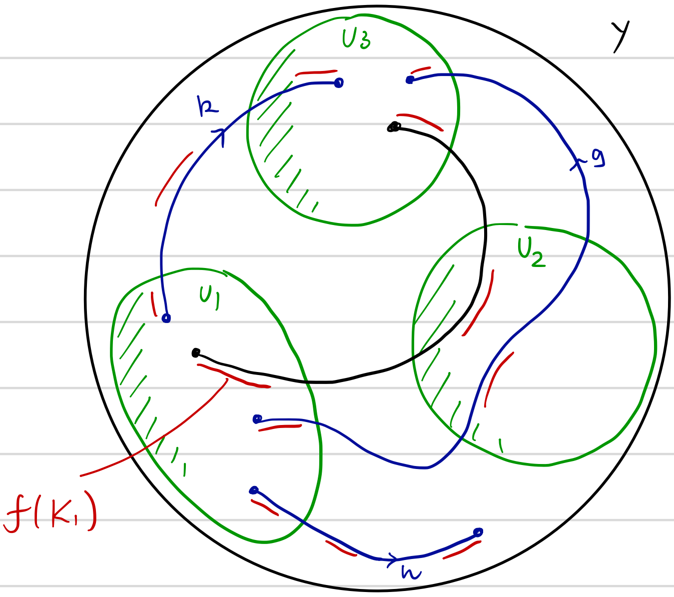
$$S(K, U) = \{\, g : [0,1] \longrightarrow Y \text{ cts} \mid g(K) \subseteq U \,\}$$

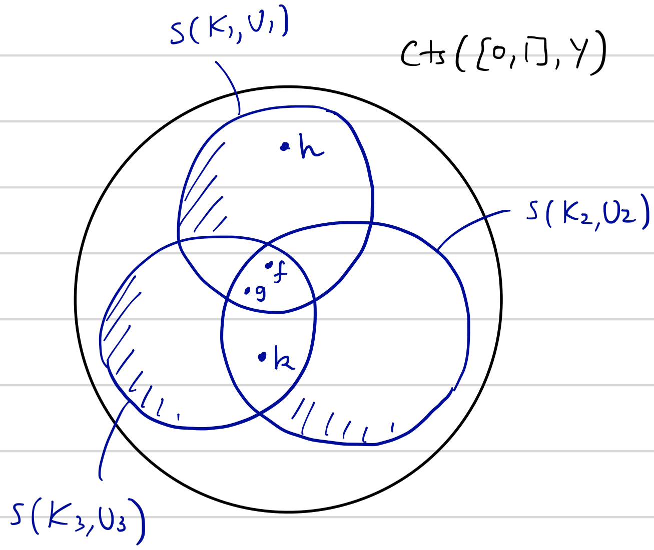


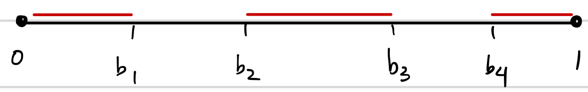


This subset is in fact open in the topology on $Cts([0,1], Y)$ as we will "discover" from first principles in lectures. More generally, if we extend the earlier example by replacing the points $\{a_1, \ldots, a_n\}$ by intervals $K_1, \ldots, K_n \subseteq [0,1]$ we can consider the open set

$$S(K_1, U_1) \cap \cdots \cap S(K_n, U_n) \subseteq Cts([0,1], Y) \qquad (*)$$







$K_1 = [0,b_1],\ K_2 = [b_2,b_3],\ K_3 = [b_4,1]$

If we allow the K_i to be arbitrary compact (i.e. closed) subsets of $[0,1]$ then the finite intersections of the form $(*)$ are a basis for the topology on path space, so the above gives you a pretty good understanding of the open sets. To make the adjunction property work, you need <u>more</u> than just the sets $S(K,U)$ for K finite, as the following shows:

$\boxed{Q1}$ The collection $\{ ev_x : Cts([0,1],Y) \longrightarrow Y \}_{x \in [0,1]}$ induces an injective continuous function

$$\Phi : Cts([0,1],Y) \longrightarrow \prod_{x\in[0,1]} Y$$

$$\Phi(f) = (f(x))_{x\in[0,1]}.$$

Is this a homeomorphism onto its image? i.e. is $Cts([0,1],Y)$ "the same as" a subspace of $\prod_{x\in[0,1]} Y$?

Recall that a basis for the product topology on $\prod_{x\in[0,1]} Y$ is given by open sets of the form $U = \prod_x U_x$ where $U_x \subseteq Y$ is open for all $x \in [0,1]$ and $\{x \in X \mid U_x \neq Y\}$ is finite. Set

$$\{x_1, \dots, x_n\} = \{x \in X \mid U_x \neq Y\}.$$

Let us identify $\prod_{x\in[0,1]} Y$ with $Y^{[0,1]}$, the set of functions $f : [0,1] \longrightarrow Y$ (not necessarily continuous). Then $f \in U$ iff. $f(x_i) \in U_{x_i}$ for $1 \le i \le n$. That is, viewed as a subset $U \subseteq Y^{[0,1]}$

$$U = \bigcap_{i=1}^{n} \{f \mid f(x_i) \in U_{x_i}\}$$

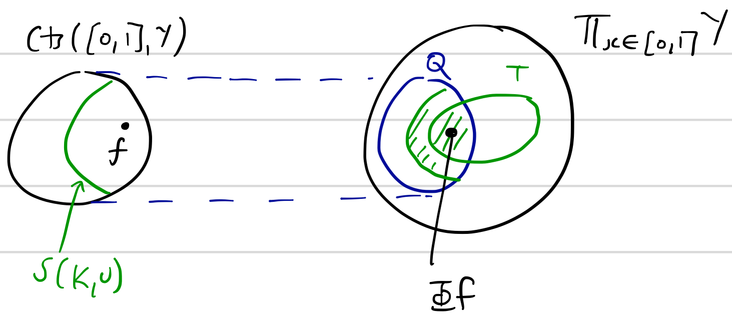
It follows that the subspace topology on $\mathrm{Cts}([0,1], Y) \subseteq Y^{[0,1]}$ inherited from the product topology has a sub-basis consisting of sets $S(K,U)$ where K is a point. Since having the adjunction property forces the $S(K,U)$'s to be open for <u>all</u> compact K, Q1 above combined with the above remarks shows that the topology on $\mathrm{Cts}([0,1], Y)$ coming from viewing this as a subspace of $Y^{[0,1]} \cong \prod_{x\in[0,1]} Y$ with the product topology <u>does not</u> satisfy the adjunction property.

$\boxed{\text{Sol}^{\text{n}}\ 1}$ Let $Q := \mathrm{Im}(\Phi)$, so that the question is, does Φ restrict to a homeomorphism $\mathrm{Cts}([0,1], Y) \cong Q$? We show that for any interval $K \subseteq [0,1]$ and open $U \subseteq Y$ which is not a <u>connected component</u> of Y (that is, for which there exists point $y \in Y \setminus U$ connected by a path to an element of U) that $\Phi(S(K,U)) \subseteq Q$ is not open, so the answer is NO.

for which there exists $y \in Y \setminus U$ and a path from u to y.

Note $S(K,U)$ is nonempty: the constant function $f : [0,1] \to Y$ sending everything to any chosen fixed $u \in U$ lies in $S(K,U)$. Supposing $\Phi(S(K,U))$ were open we would find an open neighborhood T in $\prod_{x\in[0,1]} Y$ such that

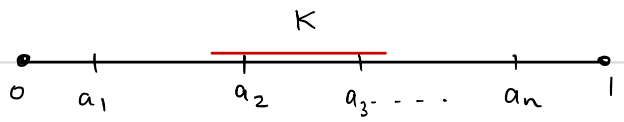
$$\Phi(f) \in T \cap Q \subseteq \Phi(S(K,U))$$
$$f \in \Phi^{-1}(T) \subseteq S(K,U)$$

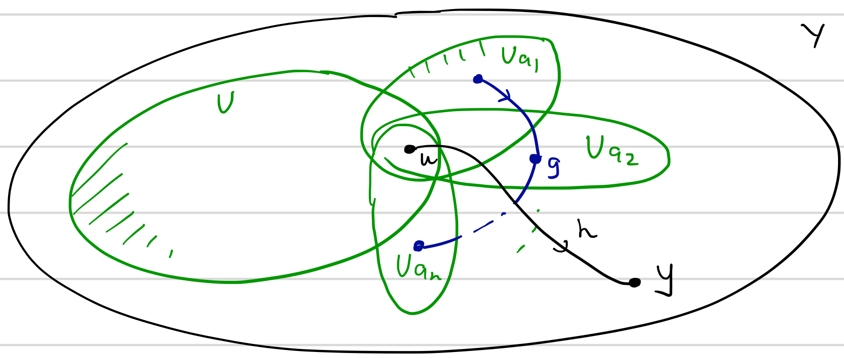


We may assume T is taken from the standard basis, i.e. there is $J \subseteq [0,1]$ finite and open sets $U_x \subseteq Y$ for all $x \in [0,1]$ such that $U_x = Y$ unless $x \in J$, and $T = \prod_{x\in[0,1]} U_x$. Write $J = \{a_1, \dots, a_n\}$ so that we have

$$\Phi^{-1}(T) = \{ g : [0,1] \to Y \text{ continuous} \mid g(a_i) \in U_{a_i} \text{ for } 1 \leq i \leq n \}$$

Note that $f \in \Phi^{-1}(T)$ so $u \in U \cap U_{a_1} \cap \dots \cap U_{a_n}$.





Now $\Phi^{-1}(T) \subseteq S(K, U)$ means that whenever $g : [0,1] \longrightarrow Y$ is continuous and satisfies $g(a_i) \in U_{a_i}$ for $1 \leq i \leq n$ (as shown) it must follow that $g(K) \subseteq U$.

Let $y \in Y \setminus U$ be connected to $u \in U$ by a path h, and let h' be the inverse path (i.e. $h'(\lambda) = h(1-\lambda)$). Let $g : [0,1] \longrightarrow Y$ be the path described informally as follows: pick a subinterval $Z \subseteq K \setminus \{a_1, \ldots, a_n\}$ not containing any a_i, and let g be constant at u for $x \notin Z$, and on Z it traverses h and then h^{-1} (so it makes an "excursion" to y in between the $a_1, \ldots, a_n$). Then $g(a_i) = u \in U_{a_i}$ so $g \in \Phi^{-1}(T)$ but clearly $g(K) \not\subseteq U$ since $g(z) = y$ for some $z \in Z$. This contradiction shows $\Phi(S(K, U))$ is not open in Q. $\square$