$\boxed{\text{Solution to Q2}}$ The empty set does not contain ∞ and is open in X so $\phi \in \mathcal{T}_{\tilde{X}}$. The set $\tilde{X}$ contains ∞ but $\tilde{X} = \phi^c \amalg \{\infty\}$ and ϕ is compact (told you we would have to care about this). Suppose $\{U_i\}_{i\in I}$ is an indexed family of open sets in $\tilde{X}$, then let $J = \{ i \in I \mid \infty \in U_i \}$ and write $(J^c = I \setminus J)$

$$\bigcup_{i\in I} U_i = \bigcup_{j\in J} U_j \cup \bigcup_{j \in J^c} U_j$$

For $j \in J$ let $K_j \subseteq X$ be compact with $U_j = K_j^c \amalg \{\infty\}$. Then, observing that U_j is open in X for $j \in J^c$, (we may assume $J \neq \phi$, otherwise it is clear)

$$\begin{aligned}\bigcup_{j\in J} U_j \cup \bigcup_{j\in J^c} U_j &= \bigcup_{j\in J} K_j^c \cup \{\infty\} \cup \bigcup_{j\in J^c} U_j \\ &= \Big(\bigcup_{j\in J} K_j^c \cup \bigcup_{j\in J^c} U_j\Big) \cup \{\infty\} \\ &= \Big(\bigcap_{j\in J} K_j \cap \bigcap_{j\in J^c} U_j^c\Big)^c \cup \{\infty\}.\end{aligned}$$

But each $K_j \subseteq X$ is compact hence closed, so $\bigcap_{j\in J} K_j \cap \bigcap_{j\in J^c} U_j^c$ is a closed subspace of a compact space K_{j_0} (pick $j_0 \in J$, and note that if $J = \phi$ there is nothing to prove) hence compact, so we are done: $\bigcup_{j\in J} U_j$ is open.

If $U, V \subseteq \tilde{X}$ are open there are four cases:

(i) $U, V \subseteq X$ are open, then $U \cap V$ is open.

(ii) $U = K^c \amalg \{\infty\}$, $V \subseteq X$ is open, $U \cap V = K^c \cap V \subseteq X$ is open.

(iii) $U \subseteq X$, $V = K^c \amalg \{\infty\}$ same as (ii).

(iv) $U = K^c \amalg \{\infty\}$, $V = L^c \amalg \{\infty\}$ with K, L compact, then
$U \cap V = (K \cup L)^c \amalg \{\infty\}$ with $K \cup L$ compact.

which shows $U \cap V$ is open in $\tilde{X}$, hence $\mathcal{T}_{\tilde{X}}$ is a topology. $\square$

$\boxed{\text{Solution to Q3}}$ Let $\{U_i\}_{i\in I}$ be an open cover of $\widetilde{X}$, and suppose $\infty \in U_{i_0}$. Then $U_{i_0} = K^c \amalg \{\infty\}$ for some compact $K \subseteq X$. The open sets $\{U_i\}_{i\in I}$ cover K, so let $J \subseteq I$ be finite with $K \subseteq \bigcup_{j\in J} U_j$. Then we have

$$\begin{aligned}
\widetilde{X} &= X \cup \{\infty\} \\
&= K \cup K^c \cup \{\infty\} \\
&= K \cup U_{i_0} \\
&\subseteq \left(\bigcup_{j\in J} U_j\right) \cup U_{i_0}
\end{aligned}$$

so $J \cup \{i_0\}$ is a finite subcover. To prove $\widetilde{X}$ is Hausdorff, it suffices to consider $x \in X$, and observe that since X is locally compact there exists $x \in U \subseteq K$ with U open, K compact, but then $x \in U$ and $\infty \in K^c \amalg \{\infty\}$ are disjoint open neighbourhoods of x, ∞ in $\widetilde{X}$. $\square$

$\boxed{\text{Solution to Q4}}$ The map $X \xrightarrow{f} \widetilde{X}$ is clearly continuous, since if $K \subseteq X$ is compact then K is closed hence K^c is open. It is a bijection onto its image, and a homeomorphism since if $U \subseteq X$ is open then $f(U)$ is open by definition. $\square$

$\boxed{\text{Solution to Q4}}$ Let $f: X \longrightarrow \mathbb{R}$ continuous be given. The data of F is just some real number $\lambda := F(\infty)$, and the fact that F is continuous is equivalent to the constraint that for all $\varepsilon > 0$ the set

$$F^{-1} B_\varepsilon(\lambda) \subseteq \widetilde{X}$$

is open. But this is equivalent to compactness of the set $\{x \in X \mid |f(x) - \lambda| \geqslant \varepsilon\}$. So we have our characterisation:

a continuous function $f : X \longrightarrow \mathbb{R}$ extends to $\widetilde{X}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $f - \lambda$ vanishes at infinity, by which we mean that for all $\varepsilon > 0$ the set $\{ x \in X \mid |f(x) - \lambda| \geqslant \varepsilon \}$ is compact.