Tutorial #5 : What is $\mathbb{R}$?

updated 1/9

What is a real number? This question is not really well-posed. A better question: what is the <u>system of real numbers</u>? What kind of thing is it? We say: it is a topological abelian group containing $\mathbb{Q}$ and in which all Cauchy sequences of rational numbers converge to a unique limit. (Your fundamental understanding of what $\mathbb{R}$ is can't hinge on a concept which pre-supposes the existence of $\mathbb{R}$! So metric spaces are a bit awkward here).

<u>Def^n</u> A sequence $(x_n)_{n=0}^{\infty}$ in a topological space X <u>converges</u> to $x \in X$ if for every open neighborhood U of x there exists $N > 0$ with $x_n \in U$ for all $n \geqslant N$.

<u>Def^n</u> A sequence $(x_n)_{n=0}^{\infty}$ in a topological abelian group A is <u>Cauchy</u> if for every open neighborhood U of 0 there exists $N > 0$ with $x_m - x_n \in U$ whenever $m, n \geqslant N$. We call A <u>complete</u> if every Cauchy sequence in A converges.

$\boxed{Q1}$ Prove that if $f : G \longrightarrow H$ is a homomorphism of topological abelian groups and $(x_n)_{n=0}^{\infty}$ is Cauchy in G then $(fx_n)_{n=0}^{\infty}$ is Cauchy in H.

$\boxed{Q2}$ Prove that a sequence in a Hausdorff space converges to at most one point.
(a top. group is Hausdorff $\Longleftrightarrow$ $\{0\}$ is closed, see Ex. L11-11(ii))

$\boxed{Q3}$ Two Cauchy sequences $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$ are <u>equivalent</u> if $(x_n - y_n)_{n=0}^{\infty}$ converges to zero. Prove this is an equivalence relation on the set of Cauchy sequences in a topological abelian group A.

We will eventually define $\mathbb{R}$ to be the set $\mathbb{Q}^c$ of Cauchy sequences in $\mathbb{Q}$ mod this rel^N.

The real numbers should be a Hausdorff topological abelian group which is complete and contains $\mathbb{Q}$ as a dense subset.

<u>Def</u>^{n} A <u>real number system</u> is a pair consisting of a complete Hausdorff topological abelian group $(\mathcal{R}, +, 0)$ and an injective homomorphism of topological groups

$$(\mathbb{Q}, +, 0) \xrightarrow{f} (\mathcal{R}, +, 0)$$

with the property that the smallest closed subset of $\mathcal{R}$ containing $f(\mathbb{Q})$ is $\mathcal{R}$ itself, and f induces a homeomorphism $\mathbb{Q} \xrightarrow{\cong} f(\mathbb{Q})$.

We prove there is (up to isomorphism) exactly one real number system (that's $\mathbb{R}$!) Note that <u>completeness</u> means any decimal expansion $0.a_1a_2a_3\cdots$ which can be viewed as a Cauchy sequence $\frac{a_1}{10}, \frac{a_1}{10}+\frac{a_2}{100}, \cdots$ in $\mathbb{Q}$, determines a unique (by Hausdorffness) element of $\mathcal{R}$.

$\boxed{\text{Q4}}$ (Uniqueness) If $(\mathcal{R}', +, 0)$ together with f' is another real number system prove there is a unique homomorphism of topological abelian groups g making the diagram

$$\begin{tikzcd}
(\mathbb{Q},+,0) \arrow[r, "f"] \arrow[dr, bend right, "f'"'] & (\mathcal{R},+,0) \arrow[d, "g"] \\
& (\mathcal{R}',+,0)
\end{tikzcd}$$

commute, and that this unique map is an <u>isomorphism</u>.

<u>We call this unique thing $\mathbb{R}$</u>

(well, we still have to prove a real number system exists)

Q5 (Existence) Let A be a topological abelian group, A^c the set of Cauchy sequences modulo equivalence. Given $U \subseteq A$ open we say a Cauchy sequence $(x_n)_{n=0}^{\infty}$ is <u>eventually in U</u> if there exists $N > 0$ such that $\forall n \geq N$ we have $x_n \in U$. A Cauchy sequence $(x_n)_{n=0}^{\infty}$ is <u>stably eventually in U</u> if every Cauchy sequence equivalent to $(x_n)_{n=0}^{\infty}$ is eventually in U. Define a subset $s(U) \subseteq A^c$ by

$$s(U) := \{ [(x_n)_{n=0}^{\infty}] \mid (x_n)_{n=0}^{\infty} \text{ is stably eventually in } U \}$$

Prove that

(i) $s(\phi) = \phi$, $s(A) = A^c$

(ii) $s(U \cap V) = s(U) \cap s(V)$

(iii) $s(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} s(U_i)$ $\qquad$ I any index set.

Hence $\{ s(U) \mid U \subseteq A \text{ open} \}$ is a topology on A^c.

<u>Note</u> To see the point of the "stably", consider $U = \{ q \in \mathbb{Q} \mid q < \pi \}$ and two Cauchy sequences converging to π, one from above and one from below. We do not want $\pi \in s(U)$ (so to speak), as $s(U)$ should be $(-\infty, \pi)$ not $(-\infty, \pi]$.

You may take for granted that A^c becomes an abelian group with the operation $[(x_n)] + [(y_n)] = [(x_n + y_n)]$, I hope you've seen this elsewhere.

Q6* A^c as defined above is a topological abelian group.

Hints If $x, y \in \mathbb{Q}$ and $x+y \in U$ then $0 \in -x-y+U$. Prove that if W is any open neighborhood of the zero element there is another open neighborhood V of 0 with $V+V \subseteq W$.

This is already ((way) more) than enough for a tutorial. But here are the remaining things to be checked. We now know $\mathbb{Q}^c$ is a topological abelian group. We need to show:

- $\mathbb{Q}^c$ is Hausdorff (use Ex L11-11).
- $\mathbb{Q}^c$ is complete (approximate any Cauchy seq. in $\mathbb{Q}^c$ by one in $\mathbb{Q}$).
- $\mathbb{Q}^c$ contains $\mathbb{Q}$ as a dense subset (easy)

This shows $\mathbb{Q}^c$ is a real number system, and we already know there is (at most) one such thing up to isomorphism, so we are entitled to give this thing a name and care about it.