Tutorial 6 : Spaces of paths

In the lectures we have started to develop the theory of <u>function spaces</u> and in this tutorial we aim to add to our intuition about these spaces with an extended study of the example of the <u>space of paths</u> in a topological space Y.

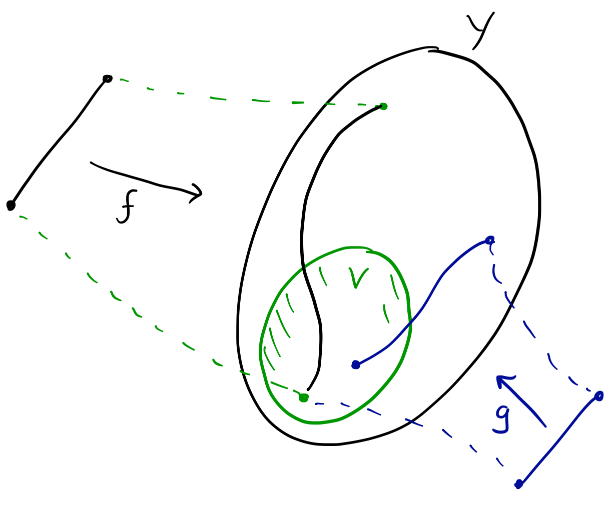
A <u>path</u> in Y is a continuous function $f : [0,1] \longrightarrow Y$. Such paths form a set $Cts([0,1], Y)$. For the purposes of this tutorial we take as given that there is a topology on this space (the <u>compact-open</u> topology) such that for all $x \in [0,1]$ the evaluation function

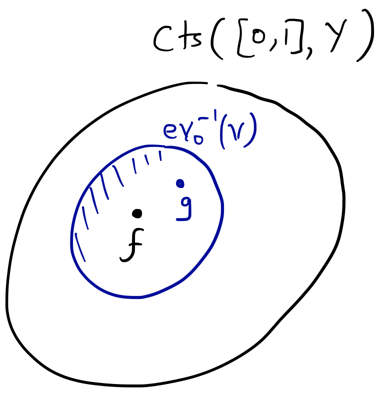
$$ev_x : Cts([0,1], Y) \longrightarrow Y$$
$$ev_x(f) = f(x)$$

is continuous. Thus <u>paths in Y</u> are the <u>points</u> of the space $Cts([0,1], Y)$. What are the open sets? Well, in particular for $V \subseteq Y$ open the subset

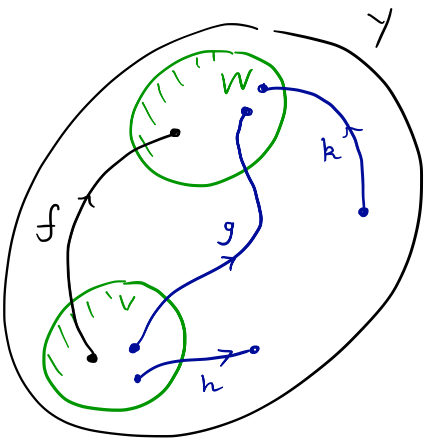
$$ev_0^{-1}(V) = \{ f : [0,1] \longrightarrow Y \text{ cts} \mid f(0) \in V \}$$

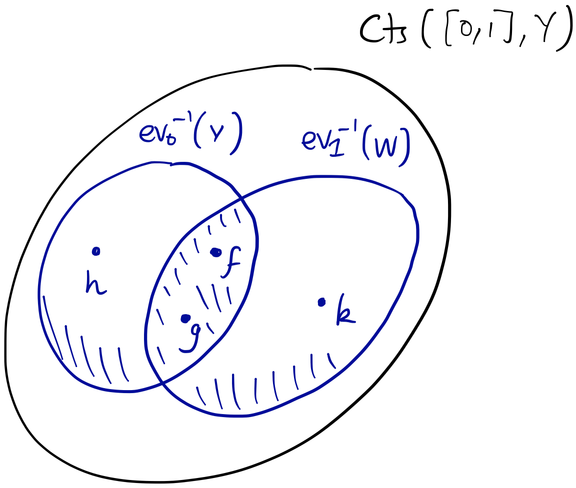
is open in the space of paths. So any particular path starting in V has an open neighborhood consisting of <u>all</u> the paths starting in V.





We can form smaller open neighborhoods of f by imposing more constraints, e.g. if $W \subseteq Y$ is open and $f(1) \in W$ then $ev_0^{-1}(V) \cap ev_1^{-1}(W)$ is a smaller neighborhood of f in $Cts([0,1], Y)$.



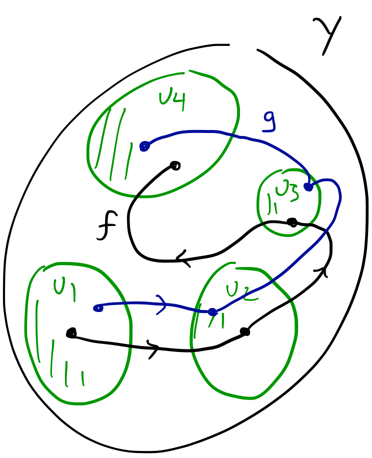


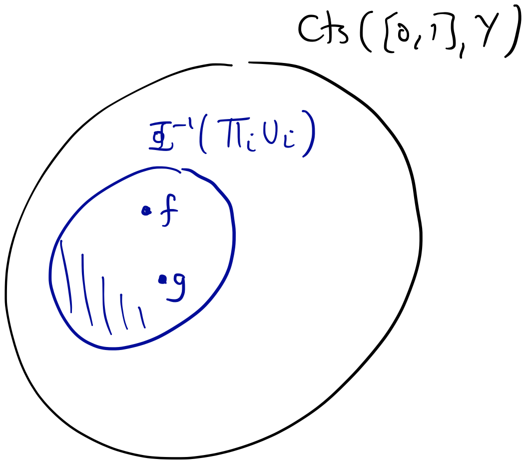
More generally, if $0 \leq a_1 < a_2 < \cdots < a_n \leq 1$ then the continuous functions $\{ev_{a_i} : Cts([0,1], Y) \longrightarrow Y\}_{i=1}^n$ induce a continuous map

$$\Phi : Cts([0,1], Y) \longrightarrow \prod_{i=1}^n Y$$

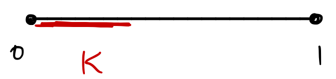
$$\Phi(f) = (f(a_1), \ldots, f(a_n))$$

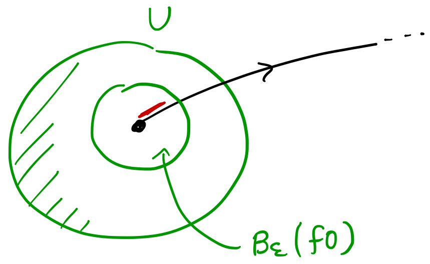
and for open sets $U_1, \ldots, U_n \subseteq Y$ the preimage $\Phi^{-1}(\prod_{i=1}^n U_i)$ is open.





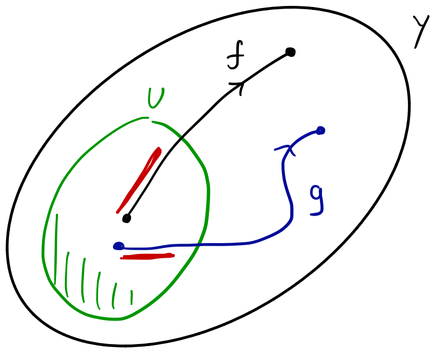
Now observe that if a path f starts in an open set $U \subseteq Y$ and Y is a metric space, there is some closed interval $K = [0, r]$ such that $f(K) \subseteq U$ (in words: if f starts in U there is some initial <u>segment</u> of f inside U. The point of this exercise is to develop intuition, so do not get hung up on the assumption Y is a metric space)

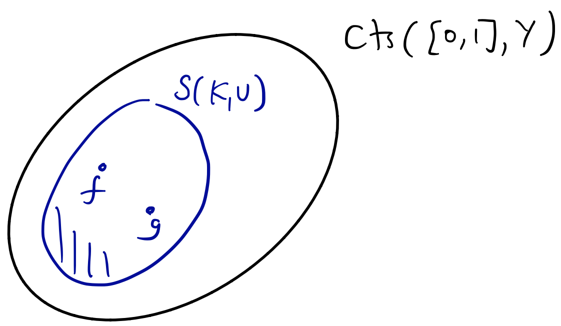




This seems like a reasonable way of prescribing paths "similar to f" which generalises the above. Rather than just impose $g(0) \in U$ (i.e. $g(K) \subseteq U$ for K a point) we can impose $g(K) \subseteq U$. Let us write (with $K = [0, r]$ as above)

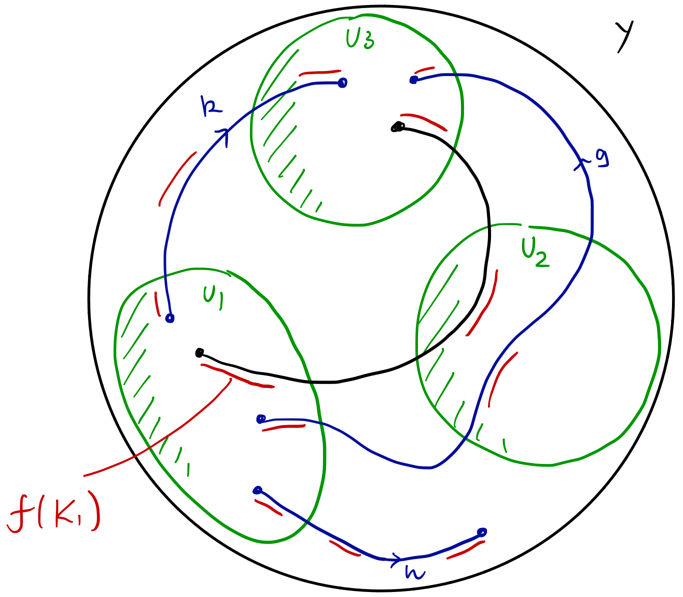
$$S(K, U) = \{ g : [0,1] \longrightarrow Y \text{ cts} \mid g(K) \subseteq U \}$$

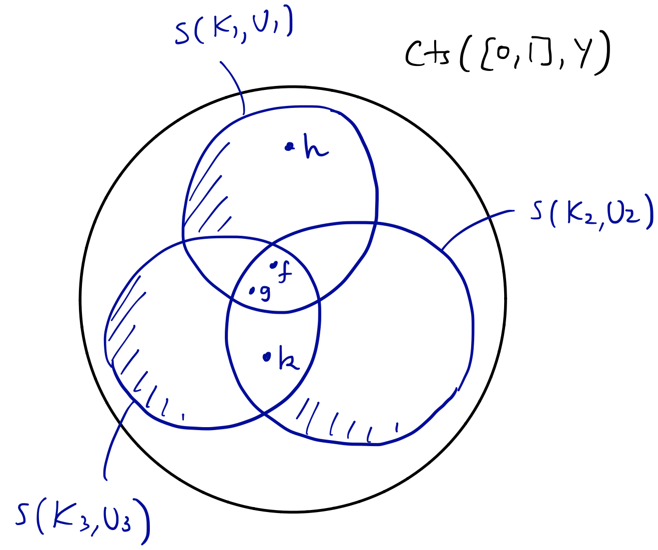


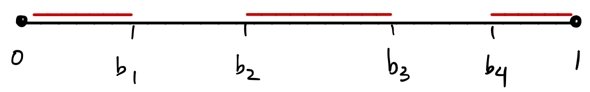


This subset is in fact open in the topology on $\mathrm{Cts}([0,1], Y)$ as we will "discover" from first principles in lectures. More generally, if we extend the earlier example by replacing the points $\{a_1, \ldots, a_n\}$ by intervals $K_1, \ldots, K_n \subseteq [0,1]$ we can consider the open set

$$S(K_1, U_1) \cap \cdots \cap S(K_n, U_n) \subseteq \mathrm{Cts}([0,1], Y) \qquad (*)$$







$K_1 = [0, b_1]$, $K_2 = [b_2, b_3]$, $K_3 = [b_4, 1]$

If we allow the K_i to be arbitrary compact (i.e. closed) subsets of $[0,1]$ then the finite intersections of the form (*) are a basis for the topology on path space, so the above gives you a pretty good understanding of the open sets. To make the adjunction property work, you need <u>more</u> than just the sets $S(K,U)$ for K finite, as the following shows:

$\boxed{Q1}$ The collection $\{ ev_x : Cts([0,1], Y) \longrightarrow Y \}_{x \in [0,1]}$ induces an injective continuous function

$$\Phi : Cts([0,1], Y) \longrightarrow \prod_{x \in [0,1]} Y$$

$$\Phi(f) = (f(x))_{x \in [0,1]}.$$

Is this a homeomorphism onto its image? i.e. is $Cts([0,1], Y)$ "the same as" a subspace of $\prod_{x \in [0,1]} Y$?