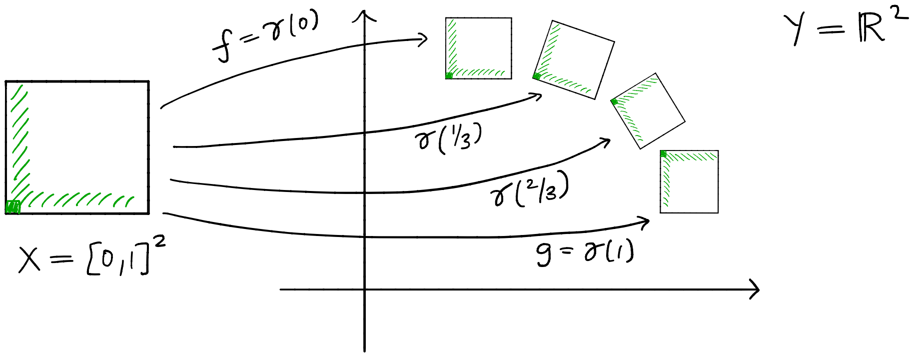
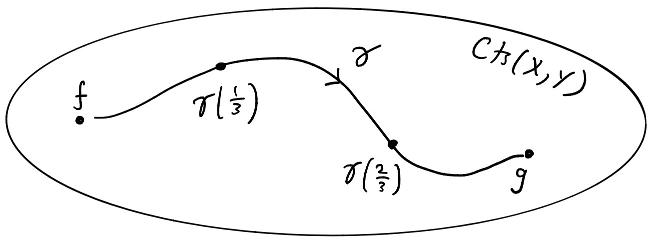
Tutorial 6 2020

In this tutorial we are going to play with the concept of <u>homotopy</u> as a way of learning how to use the adjunction property in practice. Let X, Y be locally compact Hausdorff.

<u>Def^n</u> Continuous functions $f, g : X \longrightarrow Y$ are <u>homotopic</u> if there is a path $\gamma : [0,1] \longrightarrow \mathrm{Cts}(X, Y)$ with $\gamma(0) = f$ and $\gamma(1) = g$. In this case we write $f \simeq g$.





$\boxed{Q1}$ Prove that if $X = \{*\}$ then $ev_* : \mathrm{Cts}(\{*\}, Y) \longrightarrow Y$ is a homeomorphism, and making this identification notice that two points $y, y' \in Y$ are homotopic (viewed as maps $\{*\} \longrightarrow Y$) iff. they are connected by a path in Y.

$\boxed{Q2}$ Prove that if X, Y, Z are locally compact Hausdorff and $f, g : X \longrightarrow Y$ are homotopic then

(i) if $h : Y \longrightarrow Z$ is continuous $h \circ f \simeq h \circ g$.

(ii) if $h : X \longrightarrow Y$ is continuous $f \circ h \simeq g \circ h$.

Note Ex L12-10 relates the above definition of homotopy to the more standard one.

$\boxed{Q3}$ Prove that $\simeq$ is an equivalence relation on $\mathrm{Cts}(X,Y)$.

$\boxed{Q4}$ Prove that if $Y \subseteq \mathbb{R}^n$ is convex (i.e. whenever $x, y \in Y$ then also $(1-t)x + ty \in Y$ for any $t \in [0,1]$) then any two continuous maps $f, g : X \longrightarrow Y$ are homotopic.

Solutions

$\boxed{Q1}$ The evaluation map is a composite

$$\mathrm{Cts}(\{*\},Y) \cong \mathrm{Cts}(\{*\},Y)\times\{*\} \xrightarrow{\ ev\ } Y$$

of continuous functions, hence continuous. We have a continuous function (the projection)

$$Y\times\{*\} \xrightarrow{\ F\ } Y \qquad\qquad F(y,*) = y$$

and hence under $\Psi : \mathrm{Cts}(Y\times\{*\},Y) \longrightarrow \mathrm{Cts}(Y, \mathrm{Cts}(\{*\},Y))$ this maps to continuous $\Psi(F) : Y \longrightarrow \mathrm{Cts}(\{*\},Y)$ defined by

$$\Psi(F)(y)(*) = F(y,*) = y$$

Since ev_* and $\Psi(F)$ are both continuous and mutually inverse we are done.

$\boxed{Q2}$ Suppose $f \simeq g$ so there is $\gamma : [0,1] \longrightarrow \mathrm{Cts}(X,Y)$ with $\gamma(0) = f$ and $\gamma(1) = g$. Let $h : Y \longrightarrow Z$ be given. By Lemma L12-1 the following composite is continuous

$$\begin{tikzcd}
\mathrm{Cts}(Y,Z)\times\mathrm{Cts}(X,Y) \arrow[r, "c"] & \mathrm{Cts}(X,Z) \\
\{*\}\times[0,1] \arrow[u, "\iota_h\times\gamma"'] & \\
{[0,1]} \arrow[u, phantom, "\cong" rotate=90] &
\end{tikzcd}$$

where $\iota_h : \{*\} \longrightarrow \mathrm{Cts}(Y,Z)$ is $\iota_h(*) = h$. Since this composite sends 0 to $h\circ f$ and 1 to $h\circ g$ we are done.

Q3 Reflexivity If $f \in \mathrm{Cts}(X,Y)$ then $\gamma : [0,1] \longrightarrow \mathrm{Cts}(X,Y)$ defined by $\gamma(t) = f$ for all $t \in [0,1]$ is continuous since it is a composite

$$[0,1] \longrightarrow \{*\} \xrightarrow{\iota_f} \mathrm{Cts}(X,Y)$$

of continuous maps.

Symmetry If $f \simeq g$ then let γ be a path from f to g and define γ^{rev} to be

$$[0,1] \xrightarrow{1-t} [0,1] \xrightarrow{\gamma} \mathrm{Cts}(X,Y)$$

This is continuous and sends $0 \longmapsto g$, $1 \longmapsto f$ so $g \simeq f$.

Transitivity Suppose $f \simeq g$ and $g \simeq h$ and suppose γ is a path from f to g and ρ is a path from g to h. So $\gamma, \rho : [0,1] \longrightarrow \mathrm{Cts}(X,Y)$ and $\gamma(1) = g = \rho(0)$. That is, the diagram

$$\begin{tikzcd} \{*\} \arrow[r, "\iota_1"] \arrow[d, "\iota_0"'] & {[0,1]} \arrow[d, "\gamma"] \\ {[0,1]} \arrow[r, "\rho"'] & \mathrm{Cts}(X,Y) \end{tikzcd}$$

commutes. By the universal property of the pushout there is a continuous map

$$q : [0,1] \amalg_{\{*\}} [0,1] \longrightarrow \mathrm{Cts}(X,Y)$$

restricting on the first factor to γ and on the second factor to ρ. Pre-composing q with $[0,1] \amalg_{\{*\}} [0,1] \cong [0,1]$ gives a continuous map $[0,1] \longrightarrow \mathrm{Cts}(X,Y)$ sending 0 to $\gamma(0) = f$ and 1 to $\rho(1) = h$, so $f \simeq h$.

$\boxed{Q4}$ Let $f, g \in \mathrm{Cts}(X, Y)$ be given. The path γ from f to g is

$$\gamma(t)(x) = (1-t)f(x) + t\,g(x).$$

Since Y is convex $\gamma(t)$ is a well-defined function $X \longrightarrow Y$ and clearly $\gamma(0) = f$, $\gamma(1) = g$, so it remains to show $\gamma(t)$ is continuous for all $t \in [0,1]$ and that $\gamma : [0,1] \longrightarrow \mathrm{Cts}(X,Y)$ is continuous. Observe that

$$\begin{array}{ccc}
[0,1] \times X & \xrightarrow{\binom{1-t}{t} \times \mathrm{id}_X} & [0,1] \times [0,1] \times X \\
(t, x) & (1-t, t, x) & \downarrow \mathrm{id} \times \mathrm{id} \times \binom{f}{g} \\
 & & [0,1] \times [0,1] \times Y \times Y \\
 & (1-t, t, fx, gx) & \wr\Vert \\
 & & [0,1] \times Y \times [0,1] \times Y \\
 & (1-t, fx, t, gx) & \downarrow \text{inclusion} \\
 & & (\mathbb{R} \times \mathbb{R}^n) \times (\mathbb{R} \times \mathbb{R}^n) \\
 & & \downarrow (\text{scalar-mult})^2 \\
 & ((1-t)fx, tgx) & \mathbb{R}^n \times \mathbb{R}^n \\
 & & \downarrow \text{add} \\
 & (1-t)fx + tgx & \mathbb{R}^n
\end{array}$$

is continuous (here we use that scalar multiplication and addition on $\mathbb{R}^n$ are continuous, i.e. that $\mathbb{R}^n$ is a <u>topological vector space</u>) and hence by the adjunction property

$$[0,1] \longrightarrow \mathrm{Cts}(X,Y), \quad t \longmapsto \{x \longmapsto (1-t)f(x) + tg(x)\}$$

is continuous, which is what we needed to show.