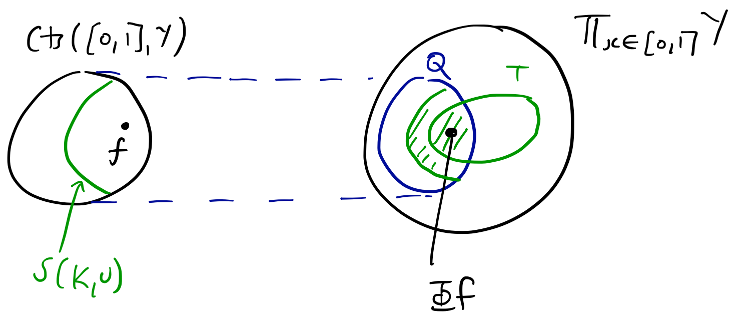
$\boxed{\text{Sol}^{\text{n}}\ 1}$ Let $Q := \mathrm{Im}(\Phi)$, so that the question is, does Φ restrict to a homeomorphism $\mathrm{Cts}([0,1], Y) \cong Q$? We show that for any interval $K \subseteq [0,1]$ and open $U \subseteq Y$ which is not a <u>connected component</u> of Y (that is, for which there exists point $y \in Y \setminus U$ connected by a path to an element of U) that $\Phi(S(K,U)) \subseteq Q$ is not open, so the answer is NO.

for which there exists $y \in Y \setminus U$ and a path from u to y.

Note $S(K,U)$ is nonempty : the constant function $f : [0,1] \to Y$ sending everything to any chosen fixed $u \in U$ lies in $S(K,U)$. Supposing $\Phi(S(K,U))$ were open we would find an open neighborhood T in $\prod_{x \in [0,1]} Y$ such that

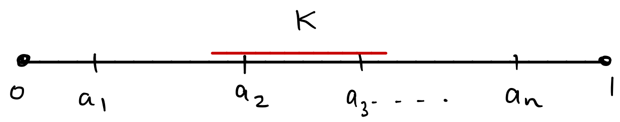
$$\Phi(f) \in T \cap Q \subseteq \Phi(S(K,U))$$
$$f \in \Phi^{-1}(T) \subseteq S(K,U)$$

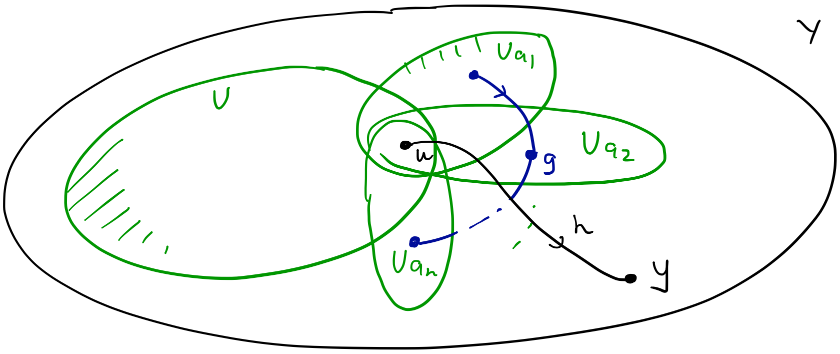


We may assume T is taken from the standard basis, i.e. there is $J \subseteq [0,1]$ finite and open sets $U_x \subseteq Y$ for all $x \in [0,1]$ such that $U_x = Y$ unless $x \in J$, and $T = \prod_{x \in [0,1]} U_x$. Write $J = \{a_1, \dots, a_n\}$ so that we have

$$\Phi^{-1}(T) = \{ g : [0,1] \to Y \text{ continuous} \mid g(a_i) \in U_{a_i} \text{ for } 1 \le i \le n \}$$

Note that $f \in \Phi^{-1}(T)$ so $u \in U \cap U_{a_1} \cap \dots \cap U_{a_n}$.





Now $\Phi^{-1}(T) \subseteq S(K,U)$ means that whenever $g : [0,1] \longrightarrow Y$ is continuous and satisfies $g(a_i) \in U_{a_i}$ for $1 \leq i \leq n$ (as shown) it must follow that $g(K) \subseteq U$.

Let $y \in Y \setminus U$ be connected to $u \in U$ by a path h, and let h' be the inverse path (i.e. $h'(\lambda) = h(1-\lambda)$). Let $g : [0,1] \longrightarrow Y$ be the path described informally as follows: pick a subinterval $Z \subseteq K \setminus \{a_1, \dots, a_n\}$ not containing any a_i, and let g be constant at u for $x \notin Z$, and on Z it traverses h and then h^{-1} (so it makes an "excursion" to y in between the $a_1, \dots, a_n$). Then $g(a_i) = u \in U_{a_i}$ so $g \in \Phi^{-1}(T)$ but clearly $g(K) \nsubseteq U$ since $g(z) = y$ for some $z \in Z$. This contradiction shows $\Phi(S(K,U))$ is not open in Q. $\square$