Tutorial 6 : Towards Fourier theory

Last week you looked at the construction of $\mathbb{R}$ from $\mathbb{Q}$, as a completion in the setting of topological abelian groups. The overall chain of constructions is

$$\mathbb{N} \xrightarrow[\text{inverses}]{\text{additive}} \mathbb{Z} \xrightarrow[\text{inverses}]{\text{multiplicative}} \mathbb{Q} \xrightarrow[\text{Cauchy sequences}]{\text{limit of}} \mathbb{R}$$

You also know $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological abelian groups, so you should find it plausible that completing $\mathbb{Q}/\mathbb{Z}$ should give $\mathbb{R}/\mathbb{Z}$ and hence S^1. But it doesn't make sense to construct $\mathbb{Q}/\mathbb{Z}$ from $\mathbb{Z}/\mathbb{Z} = 0$! So how can we extend the chain

$$?? \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow[\text{Cauchy sequences}]{\text{limit of}} \mathbb{R}/\mathbb{Z} = S^1$$

backwards to find the "finite" mathematics whose "continuum limit" is S^1? Recall that $S^1 \cong U(1)$ as topological groups, where $U(1) \subseteq \mathbb{C}$ is the multiplicative group of unit modulus complex numbers. From this point of view the obvious candidates are the subgroups of nth roots of unity

$$\begin{aligned} \mathbb{Z}/n\mathbb{Z} &\xrightarrow{\cong} U_n = \{ z \in \mathbb{C} \mid z^n = 1 \} \\ 1 &\longmapsto e^{2\pi i/n}. \end{aligned} \qquad (*)$$

$\boxed{Q1}$ (Warm-up) Prove that

$$\bigcup_{n \geq 1} U_n = \{ e^{2\pi i \theta} \mid \theta \in \mathbb{Q} \}$$

and give an isomorphism of groups $\mathbb{Q}/\mathbb{Z} \cong \bigcup_{n\geq 1} U_n$.

Q2 (i) For any two integers $m, n \geq 2$ we have $U_m \subseteq U_{mn}$. Prove that

$$\begin{tikzcd}
\mathbb{Z}/mn\mathbb{Z} \arrow[r, "\cong"] & U_{mn} \\
\mathbb{Z}/m\mathbb{Z} \arrow[u, "\varphi_{mn,m}"] \arrow[r, "\cong"'] & U_m \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

commutes, where the horizontal maps are as in $(\ast)$ above and the map $\varphi_{mn,m}$ sends $\bar{1}$ to $\bar{n}$. All arrows are group homomorphisms.

(ii) Describe the group homomorphism $\alpha_n : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ which makes the diagram below commute for all $n \geq 2$

$$\begin{tikzcd}
\mathbb{Z}/n\mathbb{Z} \arrow[r, "\alpha_n"] \arrow[d, "(\ast)"'] & \mathbb{Q}/\mathbb{Z} \arrow[d] \\
U_n \arrow[r, hook, "\text{inclusion}"'] & U(1)
\end{tikzcd}$$

this is the iso $\mathbb{Q}/\mathbb{Z} \cong \bigcup_n U_n$ you found above, composed with the inclusion to $U(1)$.

Q3 We can turn this around and use it to <u>construct</u> $\mathbb{Q}/\mathbb{Z}$ from the cyclic groups, using a categorical notion called the <u>direct limit</u>. Consider the oriented graph whose vertices are $\mathbb{Z}/m\mathbb{Z}$ for $m \geq 2$ and where we draw an edge $\mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z}$ if $m \mid k$. We label this edge with $\varphi_{k,m}$. Part of this graph (usually called a <u>diagram</u>) is

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[rr, bend left, "\varphi_{4,2}"] \arrow[rrrr, bend right=40, "\varphi_{6,2}"'] & \mathbb{Z}/3\mathbb{Z} \arrow[rrr, bend right, "\varphi_{6,3}"'] & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/5\mathbb{Z} & \mathbb{Z}/6\mathbb{Z} & \cdots
\end{tikzcd}$$

A *cocone* on this diagram is an abelian group A together with a family of homomorphisms $\{f_n : \mathbb{Z}/n\mathbb{Z} \longrightarrow A\}_{n \geq 2}$ with the property that for every pair m, k with $m | k$ we have $f_k \circ \varphi_{k,m} = f_m$. For example, all the triangles below with "green base" commute:

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[rr, bend left, "\varphi_{4,2}"] \arrow[rrrr, bend right, "\varphi_{6,2}"'] \arrow[drr, "f_2"'] & \mathbb{Z}/3\mathbb{Z} \arrow[rrr, bend right, "\varphi_{6,3}"] \arrow[dr, "f_3"] & \mathbb{Z}/4\mathbb{Z} \arrow[d, "f_4"] & \mathbb{Z}/5\mathbb{Z} \arrow[dl, "f_5"'] & \mathbb{Z}/6\mathbb{Z} \arrow[dll, "f_6"] & \cdots \\
 & & A & & &
\end{tikzcd}$$

Prove that given such a cocone there is a unique homomorphism of abelian groups $f : \mathbb{Q}/\mathbb{Z} \longrightarrow A$ such that $f \circ \alpha_n = f_n$ (α_n as in Q2) for all $n \geq 2$. We say that $\mathbb{Q}/\mathbb{Z}$ (with the α_n) is a *colimit* of the diagram (since colimits are unique, this actually completely characterises $\mathbb{Q}/\mathbb{Z}$).

$\boxed{\text{Q4}}^{*}$ Identifying the topological abelian group S^1 with $\mathbb{R}/\mathbb{Z}$ we have the canonical homomorphism of topological abelian groups $\mathbb{Q}/\mathbb{Z} \longrightarrow S^1$. Composing with α_n sits $\mathbb{Z}/n\mathbb{Z}$ as a subgroup of S^1. The circle is the "completed colimit" of all these subgroups in the following precise sense: prove that if $(A, \{f_n : \mathbb{Z}/n\mathbb{Z} \longrightarrow A\}_{n \geq 2})$ is a cocone in which A is a complete Hausdorff topological abelian group, there is a unique homomorphism of topological abelian groups $\tilde{f} : S^1 \longrightarrow A$ such that the diagram below commutes for all $n \geq 2$:

$$\begin{tikzcd}
\mathbb{Z}/n\mathbb{Z} \arrow[r, hook] \arrow[dr, "f_n"'] & S^1 \arrow[d, dashed, "\tilde{f}"] \\
 & A
\end{tikzcd}$$