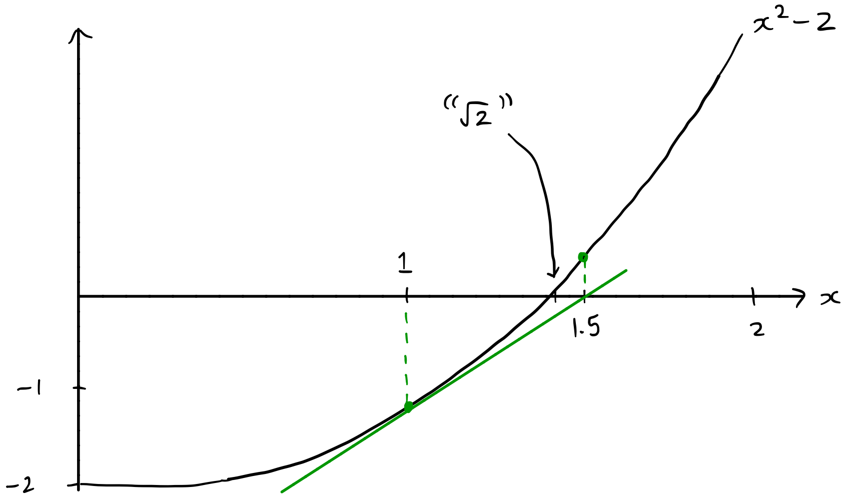
Tutorial 7 : Construction of $\mathbb{R}$

What is a real-number? To shed some light on this question consider the chain of number systems beginning at the natural numbers $\mathbb{N} = \{0,1,2,...\}$, each step in which can be viewed as adding solutions of equations:

$$\mathbb{N} \xhookrightarrow{x+2=1} \mathbb{Z} \xhookrightarrow{3x=2} \mathbb{Q} \xhookrightarrow{x^2=2} \mathbb{R} \xhookrightarrow{x^2=-1} \mathbb{C}$$

Why does $x^2=2$ "deserve" to have a solution? For one thing, we can find a sequence of approximate solutions that appears to be converging. Recall that by Newton's method we can approximate a zero of $g(x) = x^2 - 2$ by iterating the function $f(x) = x - g(x)/g'(x)$ as in the diagram (you should view $f, g : \mathbb{Q} \to \mathbb{Q}$ as functions of rational numbers and the graph below as a subset of $\mathbb{Q} \times \mathbb{Q}$)



$$f(x) = \tfrac{1}{2}x + \tfrac{1}{x}$$

$$x_0 = 1$$
$$x_1 = f(x_0) = \tfrac{3}{2}$$
$$x_2 = f(x_1) = \tfrac{17}{12} = 1.41\dot{6}$$
$$x_3 = f(x_2) = \tfrac{577}{408} \approx 1.41421568\ldots$$

$$\sqrt{2} = 1.41421356\ldots$$

The function f has a unique fixed point in $(0,\infty)$ (but not in $\mathbb{Q}$ obviously)

$$x = f(x) \iff \tfrac{1}{2}x = \tfrac{1}{x} \iff x^2 = 2.$$

So we can think of "adding $\sqrt{2}$" in one of three ways:

- adding a <u>solution</u> to the equation $x^2+2=0$
- adding a <u>fixed point</u> for $f: \mathbb{Q} \longrightarrow \mathbb{Q}$
- adding a <u>limit</u> for the sequence $(x_n)_{n=0}^{\infty} = (1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots)$.

As we will see in Lectures 14, 15 these are all closely related, but the last seems to be technically the easiest to develop (and probably also the most fundamental). To get started we need to be able to characterise when a sequence in $\mathbb{Q}$ "ought" to converge (it just lacks a limit in $\mathbb{Q}$), and we need to be able to say when two such sequences "deserving limits" are converging to the <u>same</u> "limit".

<u>Def^n</u> A sequence $(x_n)_{n=0}^{\infty}$ in a topological space X <u>converges</u> to $x \in X$ if for every open neighborhood U of x there exists $N > 0$ with $x_n \in U$ for all $n \geqslant N$.

<u>Def^n</u> A sequence $(x_n)_{n=0}^{\infty}$ in a topological abelian group A is <u>Cauchy</u> if for every open neighborhood U of 0 there exists $N > 0$ with $x_m - x_n \in U$ whenever $m, n \geqslant N$. We call A <u>complete</u> if every Cauchy sequence in A converges.

$\boxed{Q1}$ Prove that if $f: G \longrightarrow H$ is a homomorphism of topological abelian groups and $(x_n)_{n=0}^{\infty}$ is Cauchy in G then $(fx_n)_{n=0}^{\infty}$ is Cauchy in H.

$\boxed{Q2}$ Prove that a sequence in a Hausdorff space converges to at most one point.
(a top. group is Hausdorff $\Longleftrightarrow$ $\{0\}$ is closed, see Ex. L11-11(ii))

Q3 Two Cauchy sequences $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$ are <u>equivalent</u> if $(x_n - y_n)_{n=0}^{\infty}$ converges to zero. Prove this is an equivalence relation on the set of Cauchy sequences in a topological abelian group A.

Q4 Prove the sequence $(x_n)_{n=0}^{\infty} = (1, \frac{3}{2}, \frac{17}{12}, \dots)$ from earlier is Cauchy in $\mathbb{Q}$.
(Hint: use ideas of Lecture 14)

The real numbers should be a Hausdorff topological abelian group which is complete and contains $\mathbb{Q}$ as a dense subset.

<u>Def^n</u> A <u>real number system</u> is a pair consisting of a complete Hausdorff topological abelian group $(\mathcal{R}, +, 0)$ and a homomorphism of topological groups

$$(\mathbb{Q}, +, 0) \xrightarrow{f} (\mathcal{R}, +, 0)$$

which satisfies

(i) f is a homeomorphism onto its image

(ii) $\overline{f(\mathbb{Q})} = \mathcal{R}$, i.e. the smallest closed subset of $\mathcal{R}$ containing $f(\mathbb{Q})$ is $\mathcal{R}$ itself.

Note that <u>completeness</u> means any decimal expansion $0.a_1 a_2 a_3 \cdots$ which can be viewed as a Cauchy sequence $\frac{a_1}{10}, \frac{a_1}{10} + \frac{a_2}{100}, \dots$ in $\mathbb{Q}$, determines a unique (by Hausdorffness) element of $\mathcal{R}$.

$\boxed{Q5}^{**}$ (Uniqueness) If $(\mathcal{R}', +, 0)$ together with f' is another real number system prove there is a unique homomorphism of topological abelian groups g making the diagram

$$\begin{tikzcd}
(\mathbb{Q},+,0) \arrow[r, "f"] \arrow[dr, bend right, "f'"'] & (\mathcal{R},+,0) \arrow[d, "g"] \\
& (\mathcal{R}',+,0)
\end{tikzcd}$$

commute, and that this unique map is an <u>isomorphism</u>.

<u>We call this unique thing $\mathbb{R}$</u>

(well, we still have to prove a real number system exists)

$\boxed{Q6}$ (Existence) Let A be a topological abelian group, A^c the set of Cauchy sequences modulo equivalence. Given $U \subseteq A$ open we say a Cauchy sequence $(x_n)_{n=0}^\infty$ is <u>eventually in U</u> if there exists $N > 0$ such that $\forall n \geq N$ we have $x_n \in U$. A Cauchy sequence $(x_n)_{n=0}^\infty$ is <u>stably eventually in U</u> if every Cauchy sequence equivalent to $(x_n)_{n=0}^\infty$ is eventually in U. Define a subset $s(U) \subseteq A^c$ by

$$s(U) := \{ [(x_n)_{n=0}^\infty] \mid (x_n)_{n=0}^\infty \text{ is stably eventually in } U \}$$

<u>Note</u> To see the point of the "stably", consider $U = \{ q \in \mathbb{Q} \mid q < \pi \}$ and two Cauchy sequences converging to π, one from above and one from below. We do not want $\pi \in s(U)$ (so to speak), as $s(U)$ should be $(-\infty, \pi)$ not $(-\infty, \pi]$.

Prove that

(i) $s(\phi) = \phi, \; s(A) = A^c$

(ii) $s(U \cap V) = s(U) \cap s(V)$

so that $\{ s(U) \mid U \subseteq A \text{ open} \}$ is the basis for a topology on A^c.

You may take for granted that A^c becomes an abelian group with the operation $[(x_n)] + [(y_n)] = [(x_n + y_n)]$, I hope you've seen this elsewhere.

$\boxed{Q7}^*$ A^c as defined above is a topological abelian group.

Hints If $x, y \in \mathbb{Q}$ and $x + y \in U$ then $0 \in -x - y + U$. Prove that if W is any open neighborhood of the zero element there is another open neighborhood V of 0 with $V + V \subseteq W$.

The remaining things to be checked: We now know $\mathbb{Q}^c$ is a topological abelian group. We need to show:

- $\mathbb{Q}^c$ is Hausdorff (use Ex L11-11).
- $\mathbb{Q}^c$ is complete (approximate any Cauchy seq. in $\mathbb{Q}^c$ by one in $\mathbb{Q}$).
- $\mathbb{Q}^c$ contains $\mathbb{Q}$ as a dense subset (easy)

This will show $\mathbb{Q}^c$ is a real number system, and since such a thing is unique we can give it a name: the real numbers $\mathbb{R}$.