Solutions

Q1 Let U be an open neighborhood of 0 in H, so that $f^{-1}U$ is an open neighbourhood of U in G. Then let $N > 0$ be such that $x_m - x_n \in f^{-1}U$ for $m, n \geq N$ so that

$$f(x_m) - f(x_n) = f(x_m - x_n) \in U.$$

Q3 Reflexive clear

Symmetric The negation $- : A \to A$ is continuous by hypothesis, so if $(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty}$ so $(x_n - y_n)_{n=0}^{\infty}$ converges to zero, and an open neighbourhood U of 0 is given, then since $-U$ is an open neighbourhood of U we may find $N \geq 0$ s.t. for $n \geq N$

$$x_n - y_n \in -U$$

$$\text{hence} \quad y_n - x_n \in U$$

which proves $(y_n)_{n=0}^{\infty} \sim (x_n)_{n=0}^{\infty}$.

Transitive Suppose $(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty} \sim (z_n)_{n=0}^{\infty}$, and let an open neighbourhood U of 0 be given. Since $+ : A \times A \to A$ is continuous there exist $V \subseteq A$, $W \subseteq A$, both open neighbourhoods of 0, such that $V + W \subseteq U$. Let $N > 0$ be large enough so that for $n \geq N$ we have $x_n - y_n \in V$, $y_n - z_n \in W$ and hence

$$x_n - z_n = (x_n - y_n) + (y_n - z_n) \in V + W \subseteq U$$

showing $(x_n)_{n=0}^{\infty} \sim (z_n)_{n=0}^{\infty}$.

$\boxed{Q5}$ Here's some ideas to get you started...

Let $(a,b)_{\mathbb{Q}} := \{q \in \mathbb{Q} \mid a < q < b\}$, $[a,b]_{\mathbb{Q}} = \{q \in \mathbb{Q} \mid a \leq q \leq b\}$.
Then $\overline{(a,b)_{\mathbb{Q}}} = [a,b]_{\mathbb{Q}}$ and $[a,b]_{\mathbb{Q}}^{\circ} = (a,b)_{\mathbb{Q}}$. ($\overline{(-)}$ and $(-)^{\circ}$ in $\mathbb{Q}$)

We first prove that if $(\mathcal{R},+,0)$ is a real number system that any $x \in \mathcal{R}$ is the limit of a Cauchy sequence of the form $(fx_n)_{n=0}^{\infty}$ where $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$.

For $n > 0$ any integer set $U_n := (-\frac{1}{n}, \frac{1}{n})_{\mathbb{Q}}$. We want V_n to be the open subset of $\mathcal{R}$ which "is" $(-\frac{1}{n}, \frac{1}{n})$ in the usual $\mathbb{R}$. So we take V_n to be the interior (in $\mathcal{R}$) of the closure (in $\mathcal{R}$) of $f(U_n)$. Here $f : \mathbb{Q} \longrightarrow \mathcal{R}$ is given as part of the data of a real number system.

$$V_n := \overline{f(U_n)}^{\circ}$$

Now $x + V_n$ is an open neighborhood of x in $\mathcal{R}$ (as addition is continuous) and $f(\mathbb{Q}) \cap (x + V_n)$ is nonempty (for otherwise $f(\mathbb{Q}) \subseteq (x + V_n)^c$ would be a proper closed subset containing $f(\mathbb{Q})$ which is impossible. Choose some $x_n \in \mathbb{Q}$ such that $f(x_n) \in x + V_n$. We claim firstly that $(x_n)_{n=0}^{\infty}$ is Cauchy in $\mathbb{Q}$ and secondly that $f(x_n) \longrightarrow x$ in $\mathcal{R}$.

Clearly any open neighborhood of 0 in $\mathbb{Q}$ contains U_n for some n.

<u>Claim</u> If $W \subseteq \mathcal{R}$ is open then $W \cap f(\mathbb{Q})$ is dense in W (in the subspace topology).

<u>Claim</u> Any open neighborhood W of 0 in $\mathcal{R}$ contains V_n for some n.

$\boxed{Q6}$ (i) is clear,

(ii) If $U \subseteq V'$ then $s(U) \subseteq s(U')$ so $s(U \cap V) \subseteq s(U) \cap s(V)$ is immediate. For the reverse inclusion suppose an equivalence class in A^c belongs to $s(U) \cap s(V)$. Then every Cauchy sequence in this equivalence class is stably in U, and stably in V. Let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence in this equivalence class, and let $N_1 > 0$ be such that $\forall n \geqslant N_1$ $x_n \in U$, and let $N_2 > 0$ be such that $\forall n \geqslant N_2$ $x_n \in V$. Then for $n \geqslant \max\{N_1, N_2\}$, $x \in U \cap V$, so $(x_n)_{n=0}^{\infty}$ is stably in $U \cap V$.