Solutions to Tutorial 7

$\boxed{\text{Q4}}$ We begin with the commutative diagram $(A = [F]_\beta^\gamma)$

$$\begin{tikzcd}
k^n \arrow[r, "M_A"] & k^m \\
V \arrow[u, "C_\beta^{-1}", "\cong"'] \arrow[r, "F"'] & W \arrow[u, "\cong", "C_\gamma^{-1}"']
\end{tikzcd}$$

Applying $(-)^*$ to each of the linear maps and using $(f \circ g)^* = g^* \circ f^*$ we see that the bottom square in the following diagram commutes

$$\begin{tikzcd}
k^n \arrow[d, "(4.2)"'] & k^m \arrow[l, "M_{A^T}"'] \arrow[d, "(4.2)"] \arrow[dl, phantom, "(?)"] \\
(k^n)^* \arrow[d, "(C_\beta^*)^{-1}"', "\cong"] & (k^m)^* \arrow[l, "M_A^*"'] \arrow[d, "\cong"', "(C_\gamma^*)^{-1}"] \\
V^* & W^* \arrow[l, "F^*"]
\end{tikzcd}$$

Since the vertical maps are by def^N C_{β^*} and C_{γ^*} we need only show the diagram marked $(?)$ commutes. But $k^m \longrightarrow k^n \longrightarrow (k^n)^*$ sends

$$e_j \mapsto A^T e_j = \sum_{i=1}^n (A^T)_{ij} e_i = \sum_{i=1}^n A_{ji} e_i \mapsto \sum_{i=1}^n A_{ji} e_i^*$$

while $k^m \longrightarrow (k^m)^* \xrightarrow{M_A^*} (k^n)^*$ sends $e_j \longmapsto e_j^* \mapsto e_j^* \circ M_A$ and $e_j^* \circ M_A = \sum_i A_{ji} e_i^*$ since they both agree on a basis element e_i, since

$$(e_j^* \circ M_A)(e_i) = e_j^*\left(\sum_{\ell=1}^m A_{\ell i} e_\ell\right) = A_{ji}.$$

$\boxed{Q5}$

$$\begin{tikzcd}
v \arrow[rrr, dashed, bend left] \arrow[dddd, dashed] & & & Fv \arrow[ddd, dashed] \\
& V \arrow[r, "F"] \arrow[d, "\cong"] & V \arrow[d, "\cong"'] & \\
& V^* \arrow[r, "G^*"'] & V^* & \\
& & & B(Fv,-) \\
B(v,-) \arrow[rrr, dashed] & & & B(v,G(-))
\end{tikzcd}$$

$\boxed{Q6}$ First we prove

$$[v]_\beta^T\, Q\, [w]_\beta = B(v,w).$$

The left hand side and right hand side are both linear in v, w, so it is equivalent to check

$$[v_i]_\beta^T\, Q\, [v_j]_\beta = B(v_i, v_j)$$

for all i,j. But this is true by def^N, since $Q_{ij} = ([p_B]_\beta^{\beta^*})_{ij} = p_B(v_j)(v_i)$.

$$F \text{ is adjoint to } G \overset{Q5}{\iff} p_B \circ F = G^* \circ p_B$$

$$\overset{(2.1)}{\iff} [p_B \circ F]_\beta^{\beta^*} = [G^* \circ p_B]_\beta^{\beta^*}$$

$$\iff [p_B]_\beta^{\beta^*} [F]_\beta^\beta = [G^*]_{\beta^*}^{\beta^*} [p_B]_\beta^{\beta^*}$$

$$\overset{Q4}{\iff} Q\,[F]_\beta^\beta = ([G]_\beta^\beta)^T Q$$

$$\iff Q\,[F]_\beta^\beta Q^{-1} = ([G]_\beta^\beta)^T.$$