Tutorial 7 : Duality and adjoints

Dual vector spaces and the duals of linear maps will be central objects of study in the remainder of the course, and this tutorial prepares the ground with some of the basics. Throughout k is a field and vector spaces are over k, not necessarily finite-dimensional.

Def^n Given vector spaces V, W the set

$$\mathcal{L}(V,W) := \{ f : V \to W \mid f \text{ is linear} \}$$

is a vector space with the "pointwise" operations defined for $f, g \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{R}$ by

- $(f+g)(v) = f(v) + g(v)$
- $(-f)(v) = -f(v)$
- $0(v) = 0_W$. (i.e. $0 \in \mathcal{L}(V,W)$)
- $(\lambda f)(v) = \lambda \cdot f(v)$.

Def^n The dual space V^* of a vector space V is $V^* := \mathcal{L}(V,k)$. We often refer to vectors $\mu \in V^*$ as functionals.

Clearly if U, V, W are vector spaces and $f : U \to V$, $g : V \to W$ are linear then the functions given below are also linear:

$$\mathcal{L}(U,V) \xrightarrow{g \circ (-)} \mathcal{L}(U,W) \qquad h \mapsto g \circ h$$

$$\mathcal{L}(V,W) \xrightarrow{(-) \circ f} \mathcal{L}(U,W) \qquad h \mapsto h \circ f$$

Def^n If $f: U \longrightarrow V$ is linear then the linear transformation

$$V^* = \mathcal{L}(V,k) \longrightarrow \mathcal{L}(U,k) = U^*$$
$$h \longmapsto h \circ f$$

is denoted $f^*: V^* \longrightarrow U^*$.

$\boxed{Q1}$ Prove that $(1_U)^* = 1_{U^*}$ and if $f: U \to V$, $g: V \to W$ are linear then $g^* \circ f^* = (g \circ f)^*$ as linear maps $W^* \to U^*$. We say the dual is a <u>functorial</u> operation. Conclude that if f is an isomorphism then so is f^*.

$\boxed{Q2}$ Prove that $(f+g)^* = f^* + g^*$ and $(\lambda f)^* = \lambda \cdot f^*$ so that taking the dual is a linear map $(-)^*: \mathcal{L}(V,W) \longrightarrow \mathcal{L}(W^*, V^*)$.

Given $A \in M_{m,n}(k)$ let $M_A: k^n \to k^m$ be $M_A(x) = Ax$. Then there is an isomorphism of vector spaces

$$M_{m,n}(k) \longrightarrow \mathcal{L}(k^n, k^m)$$
$$A \longmapsto M_A$$

Suppose that V is <u>finite-dimensional</u> and that $\beta = (v_1, \dots, v_n)$ is a sequence of vectors in V. Then β is an ordered basis <u>if and only if</u>

$$c_\beta : k^n \longrightarrow V, \qquad c_\beta(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i v_i$$

is an isomorphism of vector spaces (it is always a well-defined linear map).

If W is another finite-dimensional vector space with ordered basis $\mathcal{C} = \{w_1, \ldots, w_m\}$ then given a linear transformation $F : V \to W$ there is a unique matrix A such that the following diagram commutes:

$$\begin{tikzcd} k^n \arrow[r, "M_A"] \arrow[d, "C_\beta"', "\cong"] & k^m \arrow[d, "C_\mathcal{C}", "\cong"'] \\ V \arrow[r, "F"'] & W \end{tikzcd} \qquad (*)$$

We call A the matrix of F with respect to $\beta, \mathcal{C}$. Some common notation:

- Given $v \in V$ we write $[v]_\beta := C_\beta^{-1}(v)$ (similarly $[w]_\mathcal{C}$).
- We write $[F]_\beta^\mathcal{C}$ for A. We may calculate from $(*)$ that the ith column of this matrix is

$$\begin{aligned}(C_\mathcal{C}^{-1} \circ F \circ C_\beta)(e_i) &= (C_\mathcal{C}^{-1} \circ F)(v_i) \\ &= C_\mathcal{C}^{-1}(F(v_i)) \\ &= [F(v_i)]_\mathcal{C}.\end{aligned}$$

$\boxed{Q3}$ Use the diagram $(*)$ to show that if $G : W \to U$ is another linear transformation and $\mathcal{D}$ is an ordered basis for U then

$$[G]_\mathcal{C}^\mathcal{D} \circ [F]_\beta^\mathcal{C} = [G \circ F]_\beta^\mathcal{D}.$$

Next we observe that a familiar matrix operation, the transpose, is "really" about dual spaces, in the following sense.

Recall from Tutorial 2 that there is a bijection

$$\left\{ \begin{matrix} \text{bilinear pairings} \\ V \times V \longrightarrow k \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{linear maps} \\ V \longrightarrow V^* \end{matrix} \right\} \qquad (4.1)$$

$$B \longmapsto \{ v \mapsto B(v,-) \}$$

If a bilinear pairing B is symmetric (i.e. $B(v,w) = B(w,v)$ for all w,v) then its associated linear map $V \longrightarrow V^*$ is an isomorphism precisely when B is <u>nondegenerate</u>, where

<u>Def^n</u> A symmetric bilinear pairing B is <u>nondegenerate</u> if

$$\forall y \in V \; B(x,y) = 0 \iff x = 0.$$

<u>Example</u> The pairing

$$B : k^n \times k^n \longrightarrow k$$
$$B(\underline{a}, \underline{b}) = \sum_{i=1}^n a_i b_i$$

is nondegenerate, and the associated isomorphism is

$$\begin{aligned} k^n &\xrightarrow{\cong} (k^n)^* \\ e_i &\longmapsto e_i^* \end{aligned} \qquad (4.2)$$

where $e_i^*(\underline{a}) = a_i$. In particular $(e_1^*, \ldots, e_n^*)$ must be a basis for $(k^n)^*$, called the <u>dual basis</u>.

<u>Def^n</u> Let V be a finite-dimensional vector space with ordered basis $\beta = (v_1, \dots, v_n)$. The <u>dual basis</u> is $\beta^* = (v_1^*, \dots, v_n^*)$ associated to the isomorphism

$$(5.1) \qquad k^n \underset{\cong}{\xrightarrow{(4.2)}} (k^n)^* \underset{\cong}{\xrightarrow{(c_\beta^*)^{-1}}} V^* \qquad \ulcorner \text{note } (c_\beta^*)^{-1} = (c_\beta^{-1})^* \lrcorner$$

which sends e_i to e_i^* and then to the functional

$$V \xrightarrow{c_\beta^{-1}} k^n \xrightarrow{e_i^*} k.$$

By def^n v_i^* is the image in V^* of e_i under (5.1), and we have just calculated that this is the functional

$$v_i^*(u) = e_i^*([u]_\beta)$$

which reads off the coefficient of v_i in the expansion of u w.r.t. β.

$\boxed{\text{Q4}}$ Suppose $V, W, \beta, \gamma, F, A$ are as in $(*)$ on p. ③. Prove that

$$\begin{tikzcd} k^n \arrow[d, "c_{\beta^*}"'] & k^m \arrow[l, "M_{A^T}"'] \arrow[d, "c_{\gamma^*}"] \\ V^* & W^* \arrow[l, "F^*"] \end{tikzcd}$$

commutes, and hence $[F^*]_{\gamma^*}^{\beta^*} = \left([F]_\beta^\gamma \right)^T$, so that taking the transpose <u>means</u> taking the dual of a linear transformation. (Hint: take the dual of the entire diagram ⊛).

<u>Adjoints</u> Let (V, B) be a pair where V is a finite-dimensional vector space and $B : V \times V \longrightarrow k$ is a nondegenerate bilinear pairing. We say a pair of linear transformations $F, G : V \longrightarrow V$ are <u>adjoint</u> with respect to B if

$$B(Fv, w) = B(v, Gw) \qquad \forall v, w \in V.$$

$\boxed{\text{Q5}}$ Prove that F, G are adjoint if and only if the diagram

$$\begin{tikzcd}
V \arrow[r, "F"] \arrow[d, "\cong"] & V \arrow[d, "\cong"'] \\
V^* \arrow[r, "G^*"'] & V^*
\end{tikzcd}$$

commutes, where the vertical maps are the isomorphisms associated to B.

$\boxed{\text{Q6}}$ Let $\beta = (v_1, \dots, v_n)$ be an ordered basis of V, $p_B : V \longrightarrow V^*$ the map associated to B and $Q := [p_B]_{\beta}^{\beta^*}$ its matrix. Prove that

$$[v]_{\beta}^{T} \, Q \, [w]_{\beta} = B(v, w) \qquad \forall v, w \in V.$$

and hence show F is adjoint to G with respect to B if and only if

$$Q \, [F]_{\beta}^{\beta} \, Q^{-1} = \left([G]_{\beta}^{\beta} \right)^{T}$$