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Tutorial 8 : Higher-order ODEs via fixed points

This tutorial walks you through the generalisation of Lecture 15 to systems of ODEs and thus to higher-order ODEs. This amounts to a solution of Ex L15-1, L15-2 of the lecture notes. But first, recall the original setup: we have an ODE $\varphi'(x) = h(x, \varphi(x))$, an initial value $\varphi(x_0) = y_0$, and we set up a contraction mapping

$$f(\varphi)(x) = y_0 + \int_{x_0}^{x} h(t, \varphi(t))dt$$

on some space of functions. Suppose φ is not a solution. Then the "error" up to some point x is (assuming φ is continuously differentiable and $\varphi(x_0) = y_0$)

$$E(\varphi)(x) = \int_{x_0}^{x} \big(\varphi'(t) - h(t, \varphi(t)) \big)dt$$

$$= \varphi(x) - \varphi(x_0) - \int_{x_0}^{x} h(t, \varphi(t))dt$$

$$= \varphi(x) - f(\varphi)(x)$$

That is, $E(\varphi) = \varphi - f(\varphi)$. As we iterate using f, say $\varphi_0, \varphi_1, \ldots$ with $\varphi_n = f^n(\varphi_0)$, this error can be written as

$$\sup\{ |E(\varphi_n)(x)| \mid x \in I \} = d_\infty(\varphi_n, f(\varphi_n)) = d_\infty(\varphi_n, \varphi_{n+1})$$

and it is easy to check by induction that $d_\infty(\varphi_n, \varphi_{n+1}) \leq \lambda^n d_\infty(\varphi_0, \varphi_1)$. Thus the error decreases exponentially fast with n. However it decreases from an initial value $d_\infty(\varphi_0, \varphi_1)$ that depends on $I = [x_0 - \delta, x_0 + \delta]$ since if $\varphi_0 \equiv y_0$ we have $d_\infty(\varphi_0, \varphi_1) = \sup\{ | \int_{x_0}^{x} h(t, y_0)dt | \mid x \in I \}$. So there is a priori some tradeoff between convergence and the size of I.

Systems of ODEs.

Consider a system of n first-order ODEs, where the $\varphi_i(x)$ are real-valued

$$\begin{aligned} \varphi_1'(x) &= h_1(x, \varphi_1(x), \ldots, \varphi_n(x)) & \varphi_1(x_0) &= y_1^0 \\ \varphi_2'(x) &= h_2(x, \varphi_1(x), \ldots, \varphi_n(x)) & \varphi_2(x_0) &= y_2^0 \\ &\vdots & &\vdots \\ \varphi_n'(x) &= h_n(x, \varphi_1(x), \ldots, \varphi_n(x)) & \varphi_n(x_0) &= y_n^0 . \end{aligned}$$

Let $\underline{h}: U \longrightarrow \mathbb{R}^n$ be continuous where $U \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ is open, then a solution of the above IVP on an interval $I \subseteq \mathbb{R}$ containing x_0 is a function $\underline{\varphi}: I \longrightarrow \mathbb{R}^n$ (whose components are the $\varphi_i(x)$) which is continuously differentiable (meaning each $\varphi_i(x)$ is so) with the property that as functions (where $\underline{\varphi}'(x) = (\varphi_1'(x), \ldots, \varphi_n'(x))$)

$$\underline{\varphi}' = \underline{h} \circ \langle 1, \underline{\varphi} \rangle, \quad \underline{\varphi}(x_0) = \underline{y}^0 = (y_1^0, \ldots, y_n^0).$$

Suppose $\alpha > 0$ exists with

$$\| \underline{h}(x, \underline{u}) - \underline{h}(x, \underline{v}) \| \leq \alpha \| \underline{u} - \underline{v} \| \quad \forall (x, \underline{u}), (x, \underline{v}) \in U$$

with $\| - \| : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $\| \underline{y} \| = \sum_{i=1}^n |y_i|$. Also assume that $(x_0, \underline{y}^0) \in U$.

$\boxed{Q1}$ Prove that there exists $\delta > 0$ such that the IVP has a unique solution on the interval $[x_0 - \delta, x_0 + \delta]$.

Higher-order ODEs

Consider an order n ODE for a single real-valued function y, in explicit form

$$y^{(n)}(x) = h(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \qquad (*)$$

with initial values $y(x_0) = y_0,\ y'(x_0) = y_0^{(1)}, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}$.
A solution of $(*)$ is a function $y(x)$ which is n-times continuously differentiable satisfying $(*)$ and having the specified initial values. Here $h : U \longrightarrow \mathbb{R}$ is a continuous function defined on an open set $U \subseteq \mathbb{R}^n$ containing the point $(x_0, y_0, \dots, y_0^{(n-1)})$.

Associated to this higher-order ODE is the system of n first-order ODEs

$$\left.\begin{array}{lrcl}
0 & \psi_0'(x) &=& \psi_1(x) \\
1 & \psi_1'(x) &=& \psi_2(x) \\
\vdots & & \vdots & \\
n-1 & \psi_{n-1}'(x) &=& h(x, \psi_0(x), \dots, \psi_{n-1}(x)).
\end{array}\right\} \quad (**)$$

In the framework of the previous page, with coordinates $x, u_0, \dots, u_{n-1}$ on $\mathbb{R}^{n+1}$, we have functions $h_0, \dots, h_{n-1} : U \longrightarrow \mathbb{R}$ where $h_i = u_{i+1}$ for $0 \le i \le n-2$ and $h_{n-1} = h$. The initial point is $(x_0, y_0, \dots, y_0^{(n-1)})$.

$\boxed{\text{Q2}}$ For any interval $I \subseteq \mathbb{R}$ prove there is a bijection between solutions y of $(*)$ on I and solutions $\underline{\psi} = (\psi_0, \dots, \psi_{n-1})$ of $(**)$ on I.

$\boxed{\text{Q3}}$ Solve the IVP $y'' = -y,\ y(0) = 0,\ y'(0) = 1$.