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Tutorial 8 : Higher-order ODEs via fixed points

This tutorial walks you through the generalisation of Lecture 15 to systems of ODEs and thus to higher-order ODEs. This amounts to a solution of Ex L15-1, L15-2 of the lecture notes. But first, recall the original setup: we have an ODE $\varphi'(x) = h(x, \varphi(x))$, an initial value $\varphi(x_0) = y_0$, and we set up a contraction mapping

$$f(\varphi)(x) = y_0 + \int_{x_0}^{x} h(t, \varphi(t))dt$$

on some space of functions. Suppose φ is not a solution. Then the "error" up to some point x is (assuming φ is continuously differentiable and $\varphi(x_0) = y_0$)

$$\begin{aligned} E(\varphi)(x) &= \int_{x_0}^{x} \big(\varphi'(t) - h(t, \varphi(t)) \big) dt \\ &= \varphi(x) - \varphi(x_0) - \int_{x_0}^{x} h(t, \varphi(t))dt \\ &= \varphi(x) - f(\varphi)(x) \end{aligned}$$

That is, $E(\varphi) = \varphi - f(\varphi)$. As we iterate using f, say $\varphi_0, \varphi_1, \dots$ with $\varphi_n = f^n(\varphi_0)$, this error can be written as

$$\sup\{ |E(\varphi_n)(x)| \mid x \in I \} = d_\infty(\varphi_n, f(\varphi_n)) = d_\infty(\varphi_n, \varphi_{n+1})$$

and it is easy to check by induction that $d_\infty(\varphi_n, \varphi_{n+1}) \leq \lambda^n d_\infty(\varphi_0, \varphi_1)$. Thus the error decreases exponentially fast with n. However it decreases from an initial value $d_\infty(\varphi_0, \varphi_1)$ that depends on $I = [x_0 - \delta, x_0 + \delta]$ since if $\varphi_0 \equiv y_0$ we have $d_\infty(\varphi_0, \varphi_1) = \sup\{ \,|\int_{x_0}^{x} h(t, y_0)dt| \mid x \in I \}$. So there is a priori some tradeoff between convergence and the size of I.

Systems of ODEs.

Consider a system of n first-order ODEs, where the $\varphi_i(x)$ are real-valued

$$\begin{aligned}
\varphi_1'(x) &= h_1(x, \varphi_1(x), \dots, \varphi_n(x)) & \varphi_1(x_0) &= y_1^0 \\
\varphi_2'(x) &= h_2(x, \varphi_1(x), \dots, \varphi_n(x)) & \varphi_2(x_0) &= y_2^0 \\
&\vdots & &\vdots \\
\varphi_n'(x) &= h_n(x, \varphi_1(x), \dots, \varphi_n(x)) & \varphi_n(x_0) &= y_n^0 .
\end{aligned}$$

Let $\underline{h}: U \longrightarrow \mathbb{R}^n$ be continuous where $U \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ is open, then a <u>solution</u> of the above IVP on an interval $I \subseteq \mathbb{R}$ containing x_0 is a function $\underline{\varphi}: I \longrightarrow \mathbb{R}^n$ (whose components are the $\varphi_i(x)$) which is continuously differentiable (meaning each $\varphi_i(x)$ is so) with the property that as functions (where $\underline{\varphi}'(x) = (\varphi_1'(x), \dots, \varphi_n'(x))$)

$$\underline{\varphi}' = \underline{h} \circ \langle 1, \underline{\varphi} \rangle, \quad \underline{\varphi}(x_0) = \underline{y}^0 = (y_1^0, \dots, y_n^0).$$

Suppose $\alpha > 0$ exists with

$$\| \underline{h}(x, \underline{u}) - \underline{h}(x, \underline{v}) \| \leq \alpha \| \underline{u} - \underline{v} \| \quad \forall (x, \underline{u}), (x, \underline{v}) \in U$$

with $\| - \| : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $\| \underline{y} \| = \sum_{i=1}^n |y_i|$. Also assume that $(x_0, \underline{y}^0) \in U$.

$\boxed{\text{Q1}}$ <u>Prove</u> that there exists $\delta > 0$ such that the IVP has a unique solution on the interval $[x_0 - \delta, x_0 + \delta]$.

Higher-order ODEs

Consider an order n ODE for a single real-valued function φ, in explicit form

$$\varphi^{(n)}(x) = h(x, \varphi(x), \varphi'(x), \dots, \varphi^{(n-1)}(x)) \qquad (*)$$

with initial values $\varphi(x_0) = y_0$, $\varphi'(x_0) = y_0^{(1)}, \dots, \varphi^{(n-1)}(x_0) = y_0^{(n-1)}$.
A <u>solution</u> of $(*)$ is a function $\varphi(x)$ which is n-times continuously differentiable satisfying $(*)$ and having the specified initial values. Here $h : U \longrightarrow \mathbb{R}$ is a continuous function defined on an open set $U \subseteq \mathbb{R}^n$ containing the point $(x_0, y_0, \dots, y_0^{(n-1)})$.

Associated to this higher-order ODE is the <u>system</u> of n first-order ODEs

$$\left.\begin{array}{lrcl} 0 & \psi_0'(x) &=& \psi_1(x) \\ 1 & \psi_1'(x) &=& \psi_2(x) \\ \vdots & & \vdots & \\ n-1 & \psi_{n-1}'(x) &=& h(x, \psi_0(x), \dots, \psi_{n-1}(x)). \end{array}\right\} \; (**)$$

In the framework of the previous page, with coordinates $x, u_0, \dots, u_{n-1}$ on $\mathbb{R}^{n+1}$, we have functions $h_0, \dots, h_{n-1} : U \longrightarrow \mathbb{R}$ where $h_i = u_{i+1}$ for $0 \leq i \leq n-2$ and $h_{n-1} = h$. The initial point is $(x_0, y_0, \dots, y_0^{(n-1)})$.

$\boxed{Q2}$ For any interval $I \subseteq \mathbb{R}$ prove there is a bijection between solutions φ of $(*)$ on I and solutions $\underline{\psi} = (\psi_0, \dots, \psi_{n-1})$ of $(**)$ on I.

$\boxed{Q3}$ Solve the IVP $\varphi'' = -\varphi$, $\varphi(0) = 0$, $\varphi'(0) = 1$.

<u>Solution Q1</u> Let $\delta > 0$ and $b_1, \dots, b_n > 0$ be such that $I \times J \subseteq U$ where $I = [x_0 - \delta, x_0 + \delta]$ and $J = \prod_{i=1}^{n} [y_i^\circ - b_i, y_i^\circ + b_i]$. Since h is continuous and $I \times J$ is compact, $\|h(x, \underline{u})\| \leq M$ for some fixed M and all $(x, \underline{u}) \in I \times J$. We may assume $\alpha\delta < 1$ and $M\delta < b_i$ for all $1 \leq i \leq n$. Define

$$f : \mathrm{Cts}(I, J) \longrightarrow \mathrm{Cts}(I, J)$$

$$f(\underline{\varphi})(x) = \left(y_i^\circ + \int_{x_0}^{x} h_i(t, \underline{\varphi}(t))\, dt \right)_{i=1}^{n}$$

We put the d_1-metric (i.e. $\|-\|$) on $J \subseteq \mathbb{R}^n$, and aim to prove that with respect to the induced d_∞-metric on $\mathrm{Cts}(I, J)$, f is a contraction mapping. Note that to check f is well-defined it suffices as before to check that $f(\underline{\varphi})(I) \subseteq J$ for all $\underline{\varphi} \in \mathrm{Cts}(I, J)$ which follows from

$$\left| \int_{x_0}^{x} h_i(t, \underline{\varphi}(t))\, dt \right| \leq \int_{x_0}^{x} | h_i(t, \underline{\varphi}(t)) |\, dt \leq M\delta < b_i .$$

To check f is a contraction, we compute

$$d_\infty(f\underline{\varphi}, f\underline{\psi}) = \sup\{ \| f(\underline{\varphi})(x) - f(\underline{\psi})(x) \| \mid x \in I \}$$

$$= \sup\{ \textstyle\sum_{i=1}^{n} \left| \int_{x_0}^{x} h_i(t, \underline{\varphi}(t))\, dt - \int_{x_0}^{x} h_i(t, \underline{\psi}(t))\, dt \right| \mid x \in I \}$$

$$\leq \sup\{ \textstyle\sum_{i=1}^{n} \int_{x_0}^{x} | h_i(t, \underline{\varphi}(t)) - h_i(t, \underline{\psi}(t)) |\, dt \mid x \in I \}$$

$$= \sup\{ \int_{x_0}^{x} \| \underline{h}(t, \underline{\varphi}(t)) - \underline{h}(t, \underline{\psi}(t)) \|\, dt \mid x \in I \}$$

$$\leq \sup\{ \int_{x_0}^{x} \alpha \| \underline{\varphi}(t) - \underline{\psi}(t) \|\, dt \mid x \in I \}$$

$$\leq \sup\{ \int_{x_0}^{x} \alpha\, d_\infty(\underline{\varphi}, \underline{\psi})\, dt \mid x \in I \} \leq \alpha\delta\, d_\infty(\underline{\varphi}, \underline{\psi}) . \square$$

<u>Solution Q2</u> Let S_*, S_{**} be the respective sets of solutions. Define

$$\Lambda : S_* \longrightarrow S_{**} \text{ by } \Lambda(\varphi) = (\varphi, \varphi', \dots, \varphi^{(n-1)})$$

This is well-defined since if $\varphi \in S_*$ it is n-times differentiable, so all derivatives up to $\varphi^{(n-1)}$ are differentiable hence continuous, and it is clear $\Lambda(\varphi) \in S_{**}$.
Define $\Theta : S_{**} \longrightarrow S_*$ by $\Theta(\psi_0, \dots, \psi_{n-1}) = \psi_0$. This is n-times continuously differentiable with $\psi_1 = \psi_0', \dots, \psi_{n-1} = \psi_0^{(n-1)}$ and

$$\psi_0^{(n)}(x) = h(x, \psi_0(x), \psi_0'(x), \dots, \psi_0^{(n-1)}(x))$$

so that $\psi_0 \in S_*$. Since clearly $\Lambda\Theta = id$, $\Theta\Lambda = id$ this completes the proof. $\square$

<u>Solution Q3</u> It is equivalent to solve the system (coordinates x, u_0, u_1 on $\mathbb{R}^3$)

$$\psi_0' = \psi_1 \qquad \psi_0(0) = 0 \qquad h_0 = u_1$$
$$\psi_1' = -\psi_0. \qquad \psi_1(0) = 1 \qquad h_1 = -u_0$$
$$x_0 = 0$$
$$\underline{y}^0 = (0,1).$$

which we may do by iterating the contraction mapping

$$f : Cts(I, J) \longrightarrow Cts(I, J)$$

$$f(\underline{\psi})(x) = \left(\int_0^x h_0(t, \psi_0(t), \psi_1(t))dt,\ 1 + \int_0^x h_1(t, \psi_0(t), \psi_1(t))dt \right)$$
$$= \left(\int_0^x \psi_1(t)dt,\ 1 - \int_0^x \psi_0(t)dt \right)$$

starting with some initial $\underline{\psi} \equiv (0,1)$. Note h is defined on all of $U = \mathbb{R}^3$, and if we choose $b_0 = b_1 = b$ then $\| \underline{h}(x,u,v) \| = |u| + |v| \leq b + 1 + b = 2b+1$.

Note that

$$\| h(x,u,v) - h(x,u',v') \| = \| (v,-u) - (v',-u') \|$$

$$= \| (v-v', u'-u) \|$$

$$= |v-v'| + |u-u'|$$

$$= \| (u,v) - (u',v') \|$$

so any $\alpha \geqslant 1$ will do. We want to take δ as large as possible with $\alpha\delta < 1$ so we take $\alpha = 1$. We take $M = 2b+1$ so that

$$b/M = b/2b+1 .$$

Taking b arbitrarily large b/M is as close to $1/2$ as we wish, and we can arrange $M\delta < b$ by taking δ to be any fixed value $\delta < 1/2$. This fixes all the hypotheses, and on $I = [-\delta, \delta]$ we compute that the following sequence converges uniformly to a unique solution:

$$\begin{aligned}
\underline{\psi}^0 &\equiv (0,1) \\
\underline{\psi}^1 &= f(\underline{\psi}^0) = (x, 1) \\
\underline{\psi}^2 &= f(\underline{\psi}^1) = (x, 1 - \tfrac{1}{2}x^2) \\
\underline{\psi}^3 &= f(\underline{\psi}^2) = (x - \tfrac{1}{2}\cdot\tfrac{1}{3}x^3, 1 - \tfrac{1}{2}x^2) \\
\underline{\psi}^4 &= f(\underline{\psi}^3) = (x - \tfrac{1}{2}\cdot\tfrac{1}{3}x^3, 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4)
\end{aligned}$$

and so clearly the limit is $\underline{\psi}^* = (\sin(x), \cos(x))$, yielding the solution $\sin(x)$ to the original IVP.