Solution Q1 Let $\delta > 0$ and $b_1, \ldots, b_n > 0$ be such that $I \times J \subseteq U$ where $I = [x_0 - \delta, x_0 + \delta]$ and $J = \prod_{i=1}^n [y_i^\circ - b_i, y_i^\circ + b_i]$. Since $h$ is continuous and $I \times J$ is compact, $\|h(x, \underline{u})\| \leq M$ for some fixed $M$ and all $(x, \underline{u}) \in I \times J$. We may assume $\alpha\delta < 1$ and $M\delta < b_i$ for all $1 \leq i \leq n$. Define

$$f : \mathrm{Cts}(I, J) \longrightarrow \mathrm{Cts}(I, J)$$

$$f(\underline{\varphi})(x) = \left( y_i^\circ + \int_{x_0}^{x} h_i(t, \underline{\varphi}(t))\, dt \right)_{i=1}^{n}$$

We put the $d_1$-metric (i.e. $\|-\|$) on $J \subseteq \mathbb{R}^n$, and aim to prove that with respect to the induced $d_\infty$-metric on $\mathrm{Cts}(I, J)$, $f$ is a contraction mapping. Note that to check $f$ is well-defined it suffices as before to check that $f(\underline{\varphi})(I) \subseteq J$ for all $\underline{\varphi} \in \mathrm{Cts}(I, J)$ which follows from

$$\left| \int_{x_0}^{x} h_i(t, \underline{\varphi}(t))\, dt \right| \leq \int_{x_0}^{x} | h_i(t, \underline{\varphi}(t)) |\, dt \leq M\delta < b_i.$$

To check $f$ is a contraction, we compute

$$d_\infty(f\underline{\varphi}, f\underline{\psi}) = \sup\{ \| f(\underline{\varphi})(x) - f(\underline{\psi})(x) \| \mid x \in I \}$$

$$= \sup\left\{ \sum_{i=1}^n \left| \int_{x_0}^{x} h_i(t, \underline{\varphi}(t))\, dt - \int_{x_0}^{x} h_i(t, \underline{\psi}(t))\, dt \right| \;\middle|\; x \in I \right\}$$

$$\leq \sup\left\{ \sum_{i=1}^n \int_{x_0}^{x} | h_i(t, \underline{\varphi}(t)) - h_i(t, \underline{\psi}(t)) |\, dt \;\middle|\; x \in I \right\}$$

$$= \sup\left\{ \int_{x_0}^{x} \| \underline{h}(t, \underline{\varphi}(t)) - \underline{h}(t, \underline{\psi}(t)) \|\, dt \;\middle|\; x \in I \right\}$$

$$\leq \sup\left\{ \int_{x_0}^{x} \alpha \| \underline{\varphi}(t) - \underline{\psi}(t) \|\, dt \;\middle|\; x \in I \right\}$$

$$\leq \sup\left\{ \int_{x_0}^{x} \alpha\, d_\infty(\underline{\varphi}, \underline{\psi})\, dt \;\middle|\; x \in I \right\} \leq \alpha\delta\, d_\infty(\underline{\varphi}, \underline{\psi}). \;\square$$

Solution Q2 Let $S_*$, $S_{**}$ be the respective sets of solutions. Define

$$\Lambda : S_* \longrightarrow S_{**} \text{ by } \Lambda(\varphi) = (\varphi, \varphi', \dots, \varphi^{(n-1)})$$

This is well-defined since if $\varphi \in S_*$ it is n-times differentiable, so all derivatives up to $\varphi^{(n-1)}$ are differentiable hence continuous, and it is clear $\Lambda(\varphi) \in S_{**}$. Define $\Theta : S_{**} \longrightarrow S_*$ by $\Theta(\psi_0, \dots, \psi_{n-1}) = \psi_0$. This is n-times continuously differentiable with $\psi_1 = \psi_0', \dots, \psi_{n-1} = \psi_0^{(n-1)}$ and

$$\psi_0^{(n)}(x) = h(x, \psi_0(x), \psi_0'(x), \dots, \psi_0^{(n-1)}(x))$$

so that $\psi_0 \in S_*$. Since clearly $\Lambda \Theta = id$, $\Theta \Lambda = id$ this completes the proof. □

Solution Q3 It is equivalent to solve the system (coordinates $x, u_0, u_1$ on $\mathbb{R}^3$)

$$\begin{aligned} \psi_0' &= \psi_1 \qquad & \psi_0(0) &= 0 \\ \psi_1' &= -\psi_0. \qquad & \psi_1(0) &= 1 \end{aligned}$$

$h_0 = u_1$

$h_1 = -u_0$

$x_0 = 0$

$\underline{y}^0 = (0,1)$.

which we may do by iterating the contraction mapping

$$f : Cts(I, J) \longrightarrow Cts(I, J)$$

$$f(\underline{\psi})(x) = \left( \int_0^x h_0(t, \psi_0(t), \psi_1(t))dt,\ 1 + \int_0^x h_1(t, \psi_0(t), \psi_1(t))dt \right)$$

$$= \left( \int_0^x \psi_1(t)dt,\ 1 - \int_0^x \psi_0(t)dt \right)$$

starting with some initial $\underline{\psi} \equiv (0,1)$. Note $h$ is defined on all of $U = \mathbb{R}^3$, and if we choose $b_0 = b_1 = b$ then $\|\underline{h}(x,u,v)\| = |u| + |v| \leq b + 1 + b = 2b + 1$.

Note that

$$\| h(x,u,v) - h(x,u',v') \| = \| (v,-u) - (v',-u') \|$$

$$= \| ( v-v', u'-u ) \|$$

$$= |v-v'| + |u-u'|$$

$$= \| (u,v) - (u',v') \|$$

so any $\alpha \geq 1$ will do. We want to take $\delta$ as large as possible with $\alpha\delta < 1$ so we take $\alpha = 1$. We take $M = 2b+1$ so that

$$b/M = b/2b+1 .$$

Taking b arbitrarily large $b/M$ is as close to $1/2$ as we wish, and we can arrange $M\delta < b$ by taking $\delta$ to be any fixed value $\delta < 1/2$. This fixes all the hypotheses, and on $I = [-\delta, \delta]$ we compute that the following sequence converges uniformly to a unique solution:

$$\begin{aligned}
\underline{\psi}^0 &\equiv (0,1) \\
\underline{\psi}^1 &= f(\underline{\psi}^0) = (x, 1) \\
\underline{\psi}^2 &= f(\underline{\psi}^1) = (x, 1 - \tfrac{1}{2}x^2) \\
\underline{\psi}^3 &= f(\underline{\psi}^2) = (x - \tfrac{1}{2}\cdot\tfrac{1}{3}x^3, 1 - \tfrac{1}{2}x^2) \\
\underline{\psi}^4 &= f(\underline{\psi}^3) = (x - \tfrac{1}{2}\cdot\tfrac{1}{3}x^3, 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4)
\end{aligned}$$

and so clearly the limit is $\underline{\psi}^* = (\sin(x), \cos(x))$, yielding the solution $\sin(x)$ to the original IVP.