# Tutorial 8 solutions

**Q3** Given an interval $I$, partition $\mathcal{P}$ of $I$ and $f : I \longrightarrow \mathbb{R}$ piece-wise constant with respect to $\mathcal{P}$, we write

$$p.c.\int_{I,\mathcal{P}} f := \sum_{J\in\mathcal{P}} a_J |J| \qquad f|_J \equiv a_J$$

Suppose for $\mathcal{P}_1 \leq \mathcal{P}_2$ with $f$ piece-wise constant w.r.t. $\mathcal{P}_2$ (hence also $\mathcal{P}_1$) we could prove that

$$p.c.\int_{I,\mathcal{P}_1} f = p.c.\int_{I,\mathcal{P}_2} f. \qquad (*)$$

Then for any pair $\mathcal{P}_1, \mathcal{P}_2$ we know $\mathcal{P}_1 \wedge \mathcal{P}_2$ is a partition and $\mathcal{P}_1 \wedge \mathcal{P}_2 \leq \mathcal{P}_i$ for $i \in \{1,2\}$ so

$$p.c.\int_{I,\mathcal{P}_1} f = p.c.\int_{I,\mathcal{P}_1\wedge\mathcal{P}_2} f = p.c.\int_{I,\mathcal{P}_2} f.$$

So it suffices to prove $(*)$. Given $J \in \mathcal{P}_2$ let $\mathcal{P}_1^J = \{K \in \mathcal{P}_1 \mid K \subseteq J\}$. This is a partition of the interval $J$, and moreover since $K \subseteq J$ we have $f|_K \equiv a_J$ for all $K \in \mathcal{P}_1^J$ (where $a_J$ is the constant value of $f$ on $J$). Hence, writing $b_K$ for the constant value of $f$ on any $K \in \mathcal{P}_1$,

$$p.c.\int_{I,\mathcal{P}_1} f = \sum_{K\in\mathcal{P}_1} b_K|K| = \sum_{\substack{J\in\mathcal{P}_2\\ K\in\mathcal{P}_1^J}} b_K|K|$$

$$= \sum_{\substack{J\in\mathcal{P}_2\\ K\in\mathcal{P}_1^J}} a_J|K| = \sum_{J\in\mathcal{P}_2} a_J\Big(\sum_{K\in\mathcal{P}_1^J}|K|\Big)$$

$$\overset{(Q2)}{=} \sum_{J\in\mathcal{P}_2} a_J|J| = p.c.\int_{I,\mathcal{P}_2} f.$$

$\boxed{Q4}$ Since $f$ is uniformly continuous (see Ex. 16-0 and its solution) we may given $\varepsilon > 0$ find $\delta > 0$ such that $|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon$. Choose $N$ an integer with $N > \frac{b-a}{\delta}$ and divide $I = [a,b]$ into $N$ equally sized subintervals $J_1, \ldots, J_N$ so that if $x, y \in J_i$ for some $i$ then

$$|x-y| \leq |J_i| = \frac{b-a}{N} < \delta$$

⌈Note that the $J_i$ are not all closed since we need to avoid overlaps, so e.g. take $\left[a, a+\frac{b-a}{N}\right)$, $\left[a+\frac{b-a}{N}, a+2\frac{b-a}{N}\right), \ldots$ $\left[b-\left(\frac{b-a}{N}\right), b\right]$⌋

so $|fx - fy| < \varepsilon$. We set

$$p_i = \sup\{ fx \mid x \in J_i \}$$
$$q_i = \inf\{ gx \mid x \in J_i \}$$

and let $g$ be the piece-wise constant function taking the values $p_i$ on $J_i$ and $h$ the function taking the values $q_i$ on $J_i$. Then $f \leq g$ and $h \leq f$ so

$$\overline{\int_I} f \leq \sum_{i=1}^{n} p_i |J_i|, \quad \underline{\int_I} f \geqslant \sum_{i=1}^{N} q_i |J_i|.$$

$$\therefore \quad \overline{\int_I} f - \underline{\int_I} f \leq \sum_{i=1}^{N} (p_i - q_i) \frac{b-a}{N}$$

Now we want to argue $p_i - q_i \leq \varepsilon$. It is tempting (but wrong) to say: well $p_i = fx$ for some $x \in J_i$ and $q_i = fy$ for some $y \in J_i$ so since $|x-y| < \delta$ we have $|p_i - q_i| = |fx - fy| < \varepsilon$. This is wrong because $J_i$ is not necessarily compact, so we cannot apply the Extreme Value Theorem. But it's OK we can argue directly: for any $x, y \in J_i$ we have $|fx - fy| < \varepsilon$ so

$$fx < fy + \varepsilon$$

$$p_i = \sup_{x \in J_i} fx \leqslant fy + \varepsilon \qquad \text{for any } y \in J_i$$

$$\therefore \quad p_i \leq \inf_{y \in J_i} fy + \varepsilon = q_i + \varepsilon.$$

Hence $p_i - q_i \leq \varepsilon$ and so

$$\overline{\int_I} f - \underline{\int_I} f \leq \sum_{i=1}^{N} \varepsilon \cdot \frac{b-a}{N} = \varepsilon \cdot (b-a)$$

which completes the proof. □