Tutorial 9 : Integration

except we do not allow empty intervals.

updated 5/10

The following is taken from T. Tao's "Analysis" Vol. 1 Ch. 11. A subset $I \subseteq \mathbb{R}$ is an <u>interval</u> if there exist $a \leq b$ with I equal to one of the following sets:

$$[a,b],\ (a,b],\ [a,b),\ (a,b).$$

In all cases we define the <u>length</u> of I to be $|I| := b-a$ (possibly zero). A <u>partition</u> of an interval I will mean a <u>finite</u> set P whose elements are pairwise disjoint intervals contained in I, whose union is all of I.

<u>Example</u> $P_1 = \{[0,1/3),[1/3,1]\}$, $P_2 = \{[0,1/2),[1/2,1]\}$ are partitions of $[0,1]$.

Given partitions P_1, P_2 of I we write $P_1 \leq P_2$ (<u>not</u> $\subseteq$) if for every $x \in P_1$ there exists $y \in P_2$ with $x \subseteq y$. This is a partial order on the set of partitions of I, and moreover given partitions P_1, P_2

$$P_1 \wedge P_2 := \{ J \cap K \mid J \in P_1, K \in P_2 \text{ and } J \cap K \neq \emptyset \}$$

is another partition of I with the property that $P_1 \wedge P_2 \leq P_i$ for $i \in \{1,2\}$ and if Q is another partition with $Q \leq P_1$, $Q \leq P_2$ then $Q \leq P_1 \wedge P_2$

$\boxed{Q1}$ Prove $\leq$ is a partial order on the set of partitions of I.

$\boxed{Q2}$ Given a partition P of I prove that $|I| = \sum_{x \in P} |x|$. (Hint: argue the statement for all pairs (I, P) by induction on the size of P).

<u>Def^n</u> Given an interval I with partition $\mathcal{P}$, a function $f: I \longrightarrow \mathbb{R}$ is <u>piecewise constant with respect to $\mathcal{P}$</u> if for all $J \in \mathcal{P}$, the function $f|_J : J \longrightarrow \mathbb{R}$ is a constant function. A function $f: I \longrightarrow \mathbb{R}$ is <u>piecewise constant</u> if it is piecewise constant with respect to <u>some</u> partition $\mathcal{P}$ of I.

$\boxed{Q3}$ Prove if $f: I \longrightarrow \mathbb{R}$ is piecewise constant with respect to partitions $\mathcal{P}_1, \mathcal{P}_2$ then

$$\sum_{J \in \mathcal{P}_1} a_J |J| = \sum_{K \in \mathcal{P}_2} b_K |K| \qquad (*)$$

where for $J \in \mathcal{P}_1$ we have $f|_J \equiv a_J$ and for $K \in \mathcal{P}_2$, $f|_K \equiv b_K$ for constants $a_J, b_K \in \mathbb{R}$. This common value $(*)$ which is independent of the partition we denote by $\text{p.c.}\int_I f$. (Hint: use $\mathcal{P}_1 \wedge \mathcal{P}_2$ to reduce to the case $\mathcal{P}_1 \subseteq \mathcal{P}_2$).

<u>Def^n</u> Let $f: I \longrightarrow \mathbb{R}$ be a bounded function on an interval I. The <u>upper Riemann integral</u> is the real number

$$\overline{\int_I} f := \inf\{ \text{p.c.}\int_I g \mid g \text{ is piecewise constant on } I \text{ and for all } x \in I, \text{ we have } g(x) \geq f(x) \}$$

while the <u>lower Riemann integral</u> is the real number

$$\underline{\int_I} f := \sup\{ \text{p.c.}\int_I g \mid g \text{ is piecewise constant on } I \text{ and for all } x \in I, \text{ we have } g(x) \leq f(x) \}.$$

If $\overline{\int_I} f = \underline{\int_I} f$ we say f is <u>Riemann integrable</u> and define

$$\int_I f := \overline{\int_I} f = \underline{\int_I} f.$$

<u>Theorem</u> Any continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.

Q4 Prove the theorem

[Long hint: aim to show that for any $\varepsilon > 0$ (with $I = [a,b]$)

$$\overline{\int_I} f - \underline{\int_I} f \leqslant \varepsilon (b-a).$$

To do this, produce a partition $J_1, \dots, J_n$ of $[a,b]$ (whose elements and length both depend on ε) and constants $p_1, \dots, p_n, q_1, \dots, q_n$ such that

$$\overline{\int_I} f \leq \sum_{i=1}^n p_i |J_i|, \qquad \underline{\int_I} f \geqslant \sum_{i=1}^n q_i |J_i|$$

so that $\overline{\int_I} f - \underline{\int_I} f \leqslant \sum_{i=1}^n (p_i - q_i)|J_i|$.

Then use that a continuous function f on $[a,b]$ is <u>uniformly</u> continuous, i.e. $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \in [a,b] (|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon)$. This is an easy consequence of compactness.]

Q5 Prove that if $f : [a,b] \to \mathbb{R}$ is continuous and $a < c < b$ then

$$\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f.$$

<u>Defn</u> A continuous function $f : X \to \mathbb{R}$ is <u>compactly supported</u> if there is $K \subseteq X$ compact such that if $x \notin K$ then $f(x) = 0$.

Q6 Prove that if $f : \mathbb{R} \to \mathbb{R}$ is compactly supported, and $[a,b]$ is such that f is zero outside $[a,b]$, then $\int_{[a,b]} f$ is independent of a, b, and we call this invariant quantity the <u>Riemann integral of f</u>.

Tutorial 8 solutions

$\boxed{Q3}$ Given an interval I, partition $\mathcal{P}$ of I and $f: I \longrightarrow \mathbb{R}$ piece-wise constant with respect to $\mathcal{P}$, we write

$$p.c.\int_{I,\mathcal{P}} f := \sum_{J\in\mathcal{P}} a_J |J| \qquad f|_J \equiv a_J$$

Suppose for $\mathcal{P}_1 \leq \mathcal{P}_2$ with f piece-wise constant w.r.t. $\mathcal{P}_2$ (hence also $\mathcal{P}_1$) we could prove that

$$p.c.\int_{I,\mathcal{P}_1} f = p.c.\int_{I,\mathcal{P}_2} f. \qquad (*)$$

Then for any pair $\mathcal{P}_1, \mathcal{P}_2$ we know $\mathcal{P}_1 \wedge \mathcal{P}_2$ is a partition and $\mathcal{P}_1 \wedge \mathcal{P}_2 \leq \mathcal{P}_i$ for $i \in \{1,2\}$ so

$$p.c.\int_{I,\mathcal{P}_1} f = p.c.\int_{I,\mathcal{P}_1\wedge\mathcal{P}_2} f = p.c.\int_{I,\mathcal{P}_2} f.$$

So it suffices to prove $(*)$. Given $J \in \mathcal{P}_2$ let $\mathcal{P}_1^J = \{K \in \mathcal{P}_1 \mid K \subseteq J\}$. This is a partition of the interval J, and moreover since $K \subseteq J$ we have $f|_K \equiv a_J$ for all $K \in \mathcal{P}_1^J$ (where a_J is the constant value of f on J). Hence, writing b_K for the constant value of f on any $K \in \mathcal{P}_1$,

$$p.c.\int_{I,\mathcal{P}_1} f = \sum_{K\in\mathcal{P}_1} b_K |K| = \sum_{\substack{J\in\mathcal{P}_2 \\ K\in\mathcal{P}_1^J}} b_K |K|$$

$$= \sum_{\substack{J\in\mathcal{P}_2 \\ K\in\mathcal{P}_1^J}} a_J |K| = \sum_{J\in\mathcal{P}_2} a_J \Big(\sum_{K\in\mathcal{P}_1^J} |K| \Big)$$

$$\overset{(Q2)}{=} \sum_{J\in\mathcal{P}_2} a_J |J| = p.c.\int_{I,\mathcal{P}_2} f.$$

Q4 Since f is uniformly continuous (see Ex. 16-O and its solution) we may given $\varepsilon > 0$ find $\delta > 0$ such that $|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon$. Choose N an integer with $N > \frac{b-a}{\delta}$ and divide $I = [a,b]$ into N equally sized subintervals $J_1, \dots, J_N$ so that if $x, y \in J_i$ for some i then

$$|x-y| \le |J_i| = \frac{b-a}{N} < \delta$$

⌜Note that the J_i are not all closed since we need to avoid overlaps, so e.g. take $\left[a, a+\frac{b-a}{N}\right)$, $\left[a+\frac{b-a}{N}, a+2\frac{b-a}{N}\right), \dots$ $\left[b-\left(\frac{b-a}{N}\right), b\right]$⌟

so $|fx - fy| < \varepsilon$. We set

$$p_i = \sup\{ fx \mid x \in J_i \}$$
$$q_i = \inf\{ gx \mid x \in J_i \}$$

and let g be the piece-wise constant function taking the values p_i on J_i and h the function taking the values q_i on J_i. Then $f \le g$ and $h \le f$ so

$$\overline{\int_I} f \le \sum_{i=1}^{n} p_i |J_i|, \quad \underline{\int_I} f \geqslant \sum_{i=1}^{N} q_i |J_i|.$$

$$\therefore \quad \overline{\int_I} f - \underline{\int_I} f \le \sum_{i=1}^{N} (p_i - q_i) \frac{b-a}{N}$$

Now we want to argue $p_i - q_i \le \varepsilon$. It is tempting (but wrong) to say: well $p_i = fx$ for some $x \in J_i$ and $q_i = fy$ for some $y \in J_i$ so since $|x-y| < \delta$ we have $|p_i - q_i| = |fx - fy| < \varepsilon$. This is wrong because J_i is not necessarily compact, so we cannot apply the Extreme Value Theorem. But it's OK we can argue directly: for any $x, y \in J_i$ we have $|fx - fy| < \varepsilon$ so

$$fx < fy + \varepsilon$$

$$p_i = \sup_{x \in J_i} fx \leqslant fy + \varepsilon \qquad \text{for any } y \in J_i$$

$$\therefore \quad p_i \le \inf_{y \in J_i} fy + \varepsilon = q_i + \varepsilon.$$

Hence $p_i - q_i \leq \varepsilon$ and so

$$\overline{\int_I} f - \underline{\int_I} f \leq \sum_{i=1}^{N} \varepsilon \cdot \frac{b-a}{N} = \varepsilon \cdot (b-a)$$

which completes the proof. $\square$