Tutorial 9: Around Stone-Weierstrass

This tutorial consists of material copied from L16, L17 which was not presented during lectures, plus solutions to some of the Exercises in those lectures. The highlight is the proof that trigonometric polynomials are dense in real-valued functions on S^1

$$\overline{\mathcal{T}\,\mathrm{Poly}(S^1,\mathbb{R})} = \mathrm{Cts}(S^1,\mathbb{R}).$$

Ultimately this fact is "responsible" for the Fourier transform for periodic functions, as we will see once we have defined the Hilbert space $L^2(S^1,\mathbb{C})$ which is the natural setting for such statements.

<u>Defn</u> An <u>$\mathbb{R}$-algebra</u> A is a vector space over $\mathbb{R}$ equipped with an additional operation $\bullet : A\times A \longrightarrow A$ (multiplication) which satisfies axioms

(i) $f(gh) = (fg)h$ for all $f,g,h\in A$ (associativity)
(ii) $\exists\, 1\in A$ s.t. $1f = f1 = f$ for all $f\in A$ (namely $1(x)\equiv 1$) (unit)
(iii) $f(g+h) = fg + fh$ for all $f,g,h\in A$. (left distributivity)
(iv) $(g+h)f = gf + hf$ for all $f,g,h\in A$. (right distributivity)
(v) $(\lambda f)g = f(\lambda g) = \lambda\cdot fg$ for all $f,g\in A$, $\lambda\in\mathbb{R}$ (bilinearity of $\bullet$)

The algebra is <u>commutative</u> if it satisfies in addition

(vi) $fg = gf$ for all $f,g\in A$

A <u>homomorphism</u> $\varphi: A\longrightarrow B$ of $\mathbb{R}$-algebras is an $\mathbb{R}$-linear map which satisfies $\varphi(1_A) = 1_B$ and $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f,g\in A$.

We know $\mathrm{Cts}(X,\mathbb{R})$ is an $\mathbb{R}$-algebra for any space X, and a <u>topological</u> $\mathbb{R}$-algebra as long as X is locally compact Hausdorff.

Claim 1 Let $j: X \longrightarrow Y$ be a continuous function. Then

$$R: Cts(Y, \mathbb{R}) \longrightarrow Cts(X, \mathbb{R}) \qquad R(f) = f \circ j$$

is a homomorphism of $\mathbb{R}$-algebras. If further X, Y are locally compact and Hausdorff, R is continuous (a homomorphism of topological $\mathbb{R}$-algebras).

Proof of claim: • $R(fg) = R(f)R(g)$ for all $f, g \in Cts(\mathbb{R}^n, \mathbb{R})$:

$$\begin{aligned} \{R(fg)\}(x) &= \{(fg) \circ j\}(x) = (fg)(j(x)) \\ &= f(j(x)) \cdot g(j(x)) \\ &= (f \circ j)(x) \cdot (g \circ j)(x) \\ &= \{R(f)R(g)\}(x) \end{aligned}$$

• $R(1) = 1$ $\qquad R(1)(x) = (1 \circ j)(x) = 1(j(x)) = 1 = 1(x).$

• $R(f+g) = R(f) + R(g)$

$$\begin{aligned} R(f+g)(x) &= \{(f+g) \circ j\}(x) = (f+g)(j(x)) \\ &= f(j(x)) + g(j(x)) = (f \circ j)(x) + (g \circ j)(x) \\ &= \{R(f) + R(g)\}(x) \end{aligned}$$

• $R(\lambda f) = \lambda R(f)$

$$\begin{aligned} R(\lambda f)(x) &= (\lambda f \circ j)(x) = (\lambda f)(j(x)) \\ &= \lambda \cdot f(j(x)) = \lambda \cdot R(f)(x) = (\lambda \cdot R(f))(x). \end{aligned}$$

The claim about continuity follows from Lemma L12-1. $\square$

<u>Def^n</u> A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is <u>polynomial</u> if there exists a function $F : \mathbb{N}^n \longrightarrow \mathbb{R}$ (where $\mathbb{N} = \{0, 1, \dots\}$) with the property that $\{ \underline{N} \in \mathbb{N}^n \mid F(\underline{N}) \neq 0 \}$ is finite and for all $x \in \mathbb{R}^n$ (write $\underline{N}$ for $(N_1, \dots, N_n)$)

$$f(x) = \sum_{\underline{N} \in \mathbb{N}^n} F(\underline{N}) \, \pi_1(x)^{N_1} \cdots \pi_n(x)^{N_n}$$

where $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the projection maps $\pi_i(x_1, \dots, x_n) = x_i$. We denote by $Poly(\mathbb{R}^n, \mathbb{R})$ the set of polynomial functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

<u>Lemma L16-1</u> Every polynomial function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, and $Poly(\mathbb{R}^n, \mathbb{R})$ is the smallest subalgebra of $Cts(\mathbb{R}^n, \mathbb{R})$ containing $\pi_1, \dots, \pi_n$. We say that $Poly(\mathbb{R}^n, \mathbb{R})$ is <u>generated</u> as an algebra by the set $\{\pi_1, \dots, \pi_n\}$.

<u>Proof</u> The polynomial function f of (4.1) may be written as

$$f = \sum_{\underline{N} \in \mathbb{N}^n} F(\underline{N}) \, \pi_1^{N_1} \cdots \pi_n^{N_n}$$

where the products (e.g. $\pi_1^{N_1} = \pi_1 \cdots \pi_1$), scalar multiplications and sums are all the algebra operations in $Cts(\mathbb{R}^n, \mathbb{R})$ as defined above. Since the set of continuous functions is <u>closed</u> under these operations (and the π_i are continuous), f must be continuous. Moreover if a subalgebra $A \subseteq Cts(\mathbb{R}^n, \mathbb{R})$ contains $\{\pi_1, \dots, \pi_n\}$ it must contain f, and the subset $Poly(\mathbb{R}^n, \mathbb{R})$ is closed under addition, multiplication and scalar multiplication (and contains 1) so it is a subalgebra, implying the second claim. $\square$

Recall: if $j : X \longrightarrow \mathbb{R}^n$ is an embedding then

$$Poly(X, j, \mathbb{R}) = \{ f \circ j \mid f \in Poly(\mathbb{R}^n, \mathbb{R}) \} \subseteq Cts(X, \mathbb{R})$$

Claim 2: If $\varphi : A \longrightarrow B$ is a homomorphism of $\mathbb{R}$-algebras, then $\varphi(A)$ is a subalgebra of B.

Proof of claim We have $1_B = \varphi(1_A) \in \varphi(A)$, and if $x, y \in \varphi(A)$, say $x = \varphi(f)$, $y = \varphi(g)$ then

$$x + y = \varphi(f) + \varphi(g) = \varphi(f+g) \in \varphi(A)$$
$$xy = \varphi(f)\varphi(g) = \varphi(fg) \in \varphi(A)$$

so $\varphi(A)$ is a subalgebra. $\square$

Exercise L16-3 Prove $\mathrm{Poly}(X, j, \mathbb{R})$ is the smallest subalgebra of $\mathrm{Cts}(X, \mathbb{R})$ containing the functions $\{\pi_1 \circ j, \ldots, \pi_n \circ j\}$.

Solution: Let $j : X \longrightarrow \mathbb{R}^n$ be an embedding. The induced map

$$R : \mathrm{Cts}(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathrm{Cts}(X, \mathbb{R}) \qquad R(f) = f \circ j \qquad (23.1)$$

is continuous by Lemma L12-1 since $\mathbb{R}^n$ is locally compact Hausdorff. By definition $\mathrm{Poly}(X, j, \mathbb{R}) = R(\mathrm{Poly}(\mathbb{R}^n, \mathbb{R}))$, and by Lemma L16-1, $\mathrm{Poly}(\mathbb{R}^n, \mathbb{R})$ is the smallest subalgebra of $\mathrm{Cts}(\mathbb{R}^n, \mathbb{R})$ containing $\{\pi_1, \ldots, \pi_n\}$. We know by Ex. 16-2 that both $\mathrm{Cts}(\mathbb{R}^n, \mathbb{R})$, $\mathrm{Cts}(X, \mathbb{R})$ are commutative $\mathbb{R}$-algebras (in fact by Lemma L16-6 they are topological $\mathbb{R}$-algebras). Moreover by Claim 1, R is a homomorphism of $\mathbb{R}$-algebras.

Note that by Claim 2, $Poly(X, j, \mathbb{R}) \subseteq Cts(X, \mathbb{R})$ is a subalgebra. It contains $\{\pi_1 \circ j, \dots, \pi_n \circ j\}$. To show it is smallest with this property let $B \subseteq Cts(X, \mathbb{R})$ be a subalgebra containing $\{\pi_1 \circ j, \dots, \pi_n \circ j\}$. Then for any (formal) polynomial $F \in \mathbb{R}[x_1, \dots, x_n]$, say

$$F = \sum_{\underline{N} \in \mathbb{N}^n} F_{\underline{N}}\, x_1^{N_1} \cdots x_n^{N_n} \qquad F_{\underline{N}} \in \mathbb{R}$$

the function

$$F(\underline{\pi} \circ j) := \sum_{\underline{N} \in \mathbb{N}^n} F_{\underline{N}} (\pi_1 \circ j)^{N_1} \cdots (\pi_n \circ j)^{N_n}$$

belongs to B, because it is obtained from the $\pi_i \circ j$ by a finite number of multiplications, scalar multiplications and additions (and for $\underline{N} = \underline{0}$ we use $1 \in B$). But we have also the element

$$F(\underline{\pi}) := \sum_{\underline{N} \in \mathbb{N}^n} F_{\underline{N}}\, \pi_1^{N_1} \cdots \pi_n^{N_n} \in Poly(\mathbb{R}^n, \mathbb{R})$$

and since R is a homomorphism of algebras

$$\begin{aligned} R(F(\underline{\pi})) &= R\Big(\textstyle\sum_{\underline{N} \in \mathbb{N}^n} F_{\underline{N}}\, \pi_1^{N_1} \cdots \pi_n^{N_n} \Big) \\ &= \textstyle\sum_{\underline{N} \in \mathbb{N}^n} R\big(F_{\underline{N}}\, \pi_1^{N_1} \cdots \pi_n^{N_n} \big) \\ &= \textstyle\sum_{\underline{N} \in \mathbb{N}^n} F_{\underline{N}}\, R\big(\pi_1^{N_1} \cdots \pi_n^{N_n} \big) \\ &= \textstyle\sum_{\underline{N} \in \mathbb{N}^n} F_{\underline{N}}\, R(\pi_1)^{N_1} \cdots R(\pi_n)^{N_n} \\ &= F(\underline{\pi} \circ j). \end{aligned}$$

We have shown $R(F(\underline{\pi})) \in B$ for any polynomial F, which shows $Poly(X, j, \mathbb{R}) \subseteq B$ as claimed. $\square$

Example L16-4 Consider the embedding

$$j : \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^2, \quad j(\theta) = (\cos\theta, \sin\theta)$$

where $\mathbb{R}/2\pi\mathbb{Z}$ is the quotient of $\mathbb{R}$ by the relation $\lambda \sim \mu$ if $\lambda - \mu \in 2\pi\mathbb{Z}$ (see Tutorial 4). We claim that $A = Poly(\mathbb{R}/2\pi\mathbb{Z}, j, \mathbb{R})$ is the smallest subalgebra of $Cts(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ containing the set $\{\cos(n\theta), \sin(n\theta)\}_{n \in \mathbb{Z}}$. By Ex. L16-3 A is the smallest subalgebra containing $\cos\theta, \sin\theta$, so the claim follows from

$$\cos(n\theta) = Re(e^{in\theta}) = Re([\cos\theta + i\sin\theta]^n) \in A$$
$$\sin(n\theta) = Im(e^{in\theta}) = Im([\cos\theta + i\sin\theta]^n) \in A$$

using the binomial formula (this does $n > 0$, but this suffices).

Def^n With $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ and j as above, we call $TPoly(S^1, \mathbb{R}) := Poly(S^1, j, \mathbb{R})$ the set of trigonometric polynomials.

Def^n We say a subalgebra $A \subseteq Cts(X, \mathbb{R})$ separates points if whenever $x, y \in X$ are distinct points there exists $f \in A$ with $f(x) \neq f(y)$.

Lemma L16-2 If $j : X \longrightarrow \mathbb{R}^n$ is an embedding then the subalgebra $Poly(X, j, \mathbb{R}) \subseteq Cts(X, \mathbb{R})$ separates points.

Proof If $x, y \in X$ are distinct, then for some $1 \leq i \leq n$ we have $\pi_i(jx) \neq \pi_i(jy)$, and so $\pi_i \circ j \in Poly(X, j, \mathbb{R})$ will do. $\square$

<u>Lemma L16-3</u> The elements of $\mathrm{TPoly}(S^1, \mathbb{R})$ are precisely the functions

$$f(\theta) = a_0 + \sum_{n=1}^{N} \big(a_n \cos(n\theta) + b_n \sin(n\theta) \big)$$

for some $a_0, a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{R}$, and $N \geq 1$. This collection of functions therefore separates points of $\mathbb{R}/2\pi\mathbb{Z}$.

<u>Proof</u> Clearly these expressions give functions in $\mathrm{Poly}(\mathbb{R}/2\pi\mathbb{Z}, j, \mathbb{R})$, so it suffices to prove functions of this form compose a <u>subalgebra</u> of $\mathrm{Cts}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$. For this it is enough to observe that these functions are closed under multiplication:

$$\begin{aligned}
\sin(mt)\cos(nt) &= \tfrac{1}{2}\big[\sin((m+n)t) + \sin((m-n)t) \big] \\
\sin(mt)\sin(nt) &= \tfrac{1}{2}\big[\cos((m-n)t) - \cos((m+n)t) \big] \\
\cos(mt)\cos(nt) &= \tfrac{1}{2}\big[\cos((m-n)t) + \cos((m+n)t) \big].
\end{aligned}$$

The claim about separating points is now immediate from Lemma L16-2. $\square$

<u>Theorem L16-3</u> (Stone-Weierstrass) Let X be a compact Hausdorff space and $A \subseteq \mathrm{Cts}(X, \mathbb{R})$ a subalgebra which separates points. Then we have $\overline{A} = \mathrm{Cts}(X, \mathbb{R})$.

<u>Corollary L16-5</u> The trigonometric polynomials are dense in $\mathrm{Cts}(S^1, \mathbb{R})$, i.e.

$$\overline{\mathrm{TPoly}(S^1, \mathbb{R})} = \mathrm{Cts}(S^1, \mathbb{R}).$$

<u>Proof</u> Again, immediate from the theorem and Lemma L16-3. $\square$