Note on Assignment 3

Here is a partial solution (just the hard bits) of Lecture 18 Exercise 5.8 (from Hilton & Stammbach p. 106).

Throughout A is an abelian group (not necessarily f.g.) and $m \in \mathbb{Z}_{>0}$. We write

$$mA = \{ma \mid a \in A\}$$
$$_mA = \{a \in A \mid ma = 0\}$$
$$A_m = A/mA.$$

The easy part is to produce exact sequences

$$0 \to \operatorname{Ext}(mA, \mathbb{Z}) \to \operatorname{Ext}(A, \mathbb{Z}) \to \operatorname{Ext}({}_mA, \mathbb{Z}) \to 0 \qquad \text{(I)}$$

$$0 \to \operatorname{Hom}(A, \mathbb{Z}) \to \operatorname{Hom}(mA, \mathbb{Z}) \to \operatorname{Ext}(A_m, \mathbb{Z}) \to \operatorname{Ext}(A, \mathbb{Z}) \to \operatorname{Ext}(mA, \mathbb{Z}) \to 0 \qquad \text{(II)}$$

and an iso $\operatorname{Hom}(A, \mathbb{Z}) \cong \operatorname{Hom}(mA, \mathbb{Z})$.

(i) $_mA = 0 \iff \operatorname{Ext}(A, \mathbb{Z})_m = 0$

<u>Proof</u> If $_mA = 0$ then $A \xrightarrow{m} A$ is mono and thus

$$0 \to A \xrightarrow{m} A \to A_m \to 0$$

is exact. The long exact Ext sequence includes

$$\cdots \to \operatorname{Ext}(A, \mathbb{Z}) \xrightarrow{m} \operatorname{Ext}(A, \mathbb{Z}) \to \underset{0}{\underset{\shortparallel}{\operatorname{Ext}^2(A_m, \mathbb{Z})}}$$

$\leftarrow$ Ext^2 always zero as gl dim $\mathbb{Z} = 1$

But this says $m\,\mathrm{Ext}(A,\mathbb{Z}) = \mathrm{Ext}(A,\mathbb{Z})$ so $\mathrm{Ext}(A,\mathbb{Z})_m = 0$.

For the converse suppose $\mathrm{Ext}(A,\mathbb{Z})_m = 0$ but $mA \neq 0$, that is, that there exist $0 \neq a \in A$ with $ma = 0$. Let $B \subseteq A$ be the cyclic subgroup generated by this a, i.e. $B = \langle a \rangle$. Clearly $B \cong \mathbb{Z}_n$ for some $n \mid m$. From

$$0 \longrightarrow B \xrightarrow{\iota} A \longrightarrow A/B \longrightarrow 0$$

we deduce a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}(A/B,\mathbb{Z}) \longrightarrow \mathrm{Ext}(A,\mathbb{Z}) \xrightarrow{\iota^*} \mathrm{Ext}(B,\mathbb{Z}) \longrightarrow \underset{(=\mathrm{Ext}^2(A/B,\mathbb{Z}))}{0}$$

Now for any abelian group C, we have $C_m \cong C \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. The point of this observation is simply that the functor $(-)_m \cong (-)\otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ is right exact. Hence we have a surjection

$$\mathrm{Ext}(A,\mathbb{Z})_m \twoheadrightarrow \mathrm{Ext}(B,\mathbb{Z})_m,$$

but by hypothesis $\mathrm{Ext}(A,\mathbb{Z})_m = 0$, whence $\mathrm{Ext}(B,\mathbb{Z})_m = 0$. But now observe

$$\begin{aligned} \mathrm{Ext}(B,\mathbb{Z}) &\cong \mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}) \\ &\cong \mathrm{Coker}\left(\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{n} \mathrm{Hom}(\mathbb{Z},\mathbb{Z})\right) \\ &\cong \mathbb{Z}_n \end{aligned}$$

use presentation $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$

and therefore $\mathrm{Ext}(B,\mathbb{Z})_m \cong (\mathbb{Z}_n)_m = \mathbb{Z}_n$. This contradiction shows $mA = 0$ as claimed.

(ii) $A_m = 0 \Rightarrow {}_m\mathrm{Ext}(A, \mathbb{Z}) = 0$.

We can do this directly: choose a projective presentation of A,

$$0 \longrightarrow R \xrightarrow{\iota} P \longrightarrow A \longrightarrow 0$$

so that

$$\mathrm{Ext}(A,\mathbb{Z}) = \mathrm{Coker}\left(\mathrm{Hom}(P,\mathbb{Z}) \xrightarrow{\iota^*} \mathrm{Hom}(R,\mathbb{Z}) \right).$$

Suppose $A_m = 0$. Then from the exact sequence

$$R_m \longrightarrow P_m \longrightarrow A_m \longrightarrow 0$$

we deduce that $R_m \longrightarrow P_m$ is surjective. Thus, for every $p \in P$ we may write

$$p = mp' + r \qquad\qquad p' \in P,\ r \in R. \qquad\qquad (*)$$

Now let $\varphi \in {}_m\mathrm{Ext}(A,\mathbb{Z})$ be given, really $[\varphi]$ where $\varphi : R \longrightarrow \mathbb{Z}$. Then $m[\varphi] = 0$ means that there is $\psi : P \longrightarrow \mathbb{Z}$ and a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[d, "m\varphi"'] & P \arrow[dl, bend left, "\psi"] \\ \mathbb{Z} & \end{tikzcd}$$

We claim $[\varphi] = 0$, that is, that we may find $\varphi' : P \to \mathbb{Z}$ with $\varphi' \circ \iota = \varphi$.

We define, using ⊛,

coming up with this is the fun part

$$\varphi'(p) = -\psi(p') + \varphi(r)$$

To check this is well-defined, suppose

$$mp_1' + r_1 = p = mp_2' + r_2$$

Then $m(p_2' - p_1') = r_1 - r_2 \in R$, so

$$\begin{aligned}\psi(mp_2') - \psi(mp_1') &= \psi(m(p_2' - p_1')) \\ &= \psi(r_1 - r_2) \\ &= (m\varphi)(r_1 - r_2)\end{aligned}$$

Hence

$$m\left[\left(-\psi(p_1') + \varphi(r_1)\right) - \left(-\psi(p_2') + \varphi(r_2)\right)\right]$$

$$= \psi(m(p_2' - p_1')) + m\varphi(r_1 - r_2)$$

$$= 0$$

But $\mathbb{Z}$ is torsion-free, so this implies the term inside the brackets is zero, and thus that φ' is well-defined. It is easy to check it is a homomorphism and $\varphi' \circ \iota = \varphi$, which shows $[\varphi] = 0$ and thus

$$_m\mathrm{Ext}(A, \mathbb{Z}) = 0.$$

(iii) if ${}_m\mathrm{Ext}(A,\mathbb{Z}) = 0 = \mathrm{Hom}(A,\mathbb{Z})$ then $A_m = 0$.

We know $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for $\underline{Ab}$, so

$$A_m = 0 \iff \mathrm{Hom}(A_m, \mathbb{Q}/\mathbb{Z}) = 0.$$

But applying $\mathrm{Hom}(A_m, \mathbb{Q}/\mathbb{Z})$ to the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an exact sequence

$$\underbrace{\mathrm{Hom}(A_m,\mathbb{Q})}_{\substack{= \\ 0 \\ (A_m \text{ is torsion})}} \longrightarrow \mathrm{Hom}(A_m,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Ext}(A_m,\mathbb{Z}) \longrightarrow \underbrace{\mathrm{Ext}(A_m,\mathbb{Q})}_{\substack{= \\ 0 \\ (\mathbb{Q} \text{ is injective})}}$$

We infer that $\mathrm{Hom}(A_m,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}(A_m,\mathbb{Z})$. Now using the long exact sequence (II) and $\mathrm{Hom}(A,\mathbb{Z}) = 0$ we have exact

$$0 \longrightarrow \mathrm{Hom}(mA,\mathbb{Z}) \longrightarrow \mathrm{Ext}(A_m,\mathbb{Z}) \longrightarrow \mathrm{Ext}(A,\mathbb{Z})$$

Then, as ${}_m(-)$ is left exact (it is $\mathrm{Hom}(\mathbb{Z}_m, -)$) we have exact

$$0 \longrightarrow {}_m\mathrm{Hom}(mA,\mathbb{Z}) \longrightarrow {}_m\mathrm{Ext}(A_m,\mathbb{Z}) \longrightarrow {}_m\mathrm{Ext}(A,\mathbb{Z}).$$

Now, two observations:

- $A_m \xrightarrow{m} A_m$ is zero, hence so is $\mathrm{Ext}(A_m,\mathbb{Z}) \xrightarrow{m} \mathrm{Ext}(A_m,\mathbb{Z})$, i.e. ${}_m\mathrm{Ext}(A_m,\mathbb{Z}) = \mathrm{Ext}(A_m,\mathbb{Z})$.

- ${}_m\mathrm{Hom}(mA,\mathbb{Z}) \cong {}_m\mathrm{Hom}(A,\mathbb{Z})$.

so we actually have an exact sequence

$$0 \longrightarrow {}_m\mathrm{Hom}(A,\mathbb{Z}) \longrightarrow \mathrm{Ext}(A_m,\mathbb{Z}) \longrightarrow {}_m\mathrm{Ext}(A,\mathbb{Z})$$

By hypothesis the first and third terms are zero. Hence $\mathrm{Ext}(A_m,\mathbb{Z})=0$ and we have already agreed this forces $A_m=0$.