MAST90068 – Lecture 1

The subjects of this course are

- Categories and functors
- Homological algebra (inc. group cohomology)
- Noncommutative algebra

You are assumed to have a working knowledge of linear algebra, group theory and rings/modules. We begin with an introduction to <u>category theory</u>, and the purpose of this first lecture is twofold: on the one hand to give the basic definitions, and on the other hand to explain <u>why</u> category theory is useful. As Arthur C. Clarke said:

> "Any sufficiently advanced technology ✓ for abstraction
> is indistinguishable from ~~magic~~ category theory"

<u>Example 1</u> Let X, Y, Z be topological spaces and suppose π, θ are continuous:

$$\begin{array}{ccc} & & X \\ & & \downarrow \pi \\ Y & \xrightarrow{\theta} & Z \end{array} \tag{1.1}$$

The <u>fiber product</u> of X, Y over Z (via π, θ), denoted $X \times_Z Y$, is

$$X \times_Z Y := \{ (x,y) \in X \times Y \mid \pi(x) = \theta(y) \} \tag{1.2}$$

with the subspace topology. There are obvious maps (the projections)

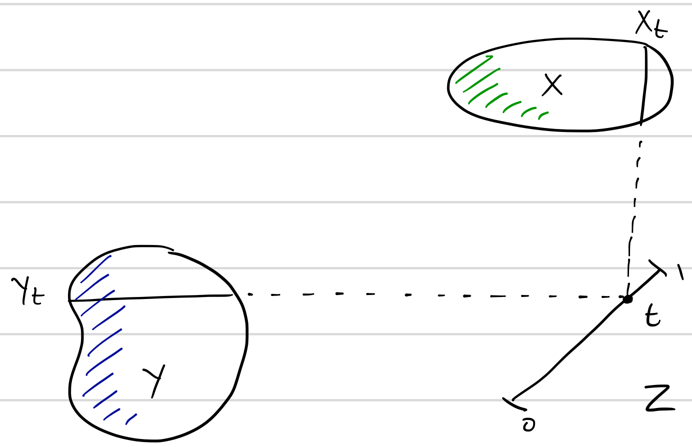
$$p_X : X \times_Z Y \longrightarrow X, \quad p_Y : X \times_Z Y \longrightarrow Y \tag{1.3}$$

which make the following diagram commute (meaning: $\pi p_X = \theta p_Y$)

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\theta"'] & Z \end{tikzcd} \qquad (2.1)$$

For instance if $Z = \{\bullet\}$ then $X \times_Z Y = X \times Y$, if $Z = [0,1]$ and we write $X_t := \pi^{-1}(t)$, $Y_t = \theta^{-1}(t)$ for the fibers over $0 \le t \le 1$ then (as sets)

$$X = \coprod_{0 \le t \le 1} X_t, \quad Y = \coprod_{0 \le t \le 1} Y_t \qquad X \times_Z Y = \coprod_{0 \le t \le 1} X_t \times Y_t$$



(2.2)

So $X \times_Z Y$ is the "fiber-wise" product of X, Y over Z.

<u>Now</u> Suppose we have $\rho : V \longrightarrow Y$ another continuous map, and form $V \times_Y (X \times_Z Y)$, as in the diagram, with projections q_{XY}, q_V

$$\begin{tikzcd} V \times_Y (X \times_Z Y) \arrow[r, "q_{XY}"] \arrow[d, "q_V"'] & X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "\pi"] \\ V \arrow[r, "\rho"'] & Y \arrow[r, "\theta"'] & Z \end{tikzcd} \qquad (2.3)$$

<u>Lemma</u> There is a homeomorphism

$$f : V\times_Y (X\times_Z Y) \xrightarrow{\ \cong\ } V\times_Z X \qquad (3.1)$$

with the second fiber product using $(\pi, \theta\rho)$.

<u>Proof</u> Recall

$$\begin{aligned} V\times_Y(X\times_Z Y) &= \{(v,u)\in V\times(X\times_Z Y) \mid p_Y(u)=\rho(v)\}\\ &= \{(v,x,y)\in V\times X\times Y \mid p_Y(x,y)=\rho(v),\ \pi(x)=\theta(y)\}\\ &= \{(v,x,y)\in V\times X\times Y \mid y=\rho(v),\ \pi(x)=\theta(y)\}\end{aligned}$$

$$\begin{array}{ccccc} (v,x,y) & & & & (v,x,\rho(v))\\ \big\downarrow & & \Big\downarrow\ \Big\uparrow & & \big\uparrow \\ (v,x) & & & & (v,x)\end{array} \qquad (3.2)$$

$$V\times_Z X = \{(v,x)\in V\times X \mid \pi(x)=\theta\rho(v)\}. \quad \square$$

<u>Example 2</u> Let X, Y, Z be abelian groups and suppose π, θ are group homomorphisms

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y \arrow[r, "\theta"'] & Z \end{tikzcd} \qquad (3.3)$$

The <u>fiber product</u> of X, Y over Z (via π, θ), denoted $X\times_Z Y$, is

$$X\times_Z Y := \{(x,y)\in X\times Y \mid \pi(x)=\theta(y)\}. \qquad (3.4)$$

This is a subgroup, therefore an abelian group, and the projections p_X, p_Y are homomorphisms.

You can check that there is an isomorphism of groups, given another homomorphism $\rho : V \longrightarrow Y$, as before:

iso of groups

$$f : V \times_Y (X \times_Z Y) \xrightarrow{\cong} V \times_Z X \qquad (4.1)$$

$$f(v, x, y) = (v, x).$$

<u>The point</u>: This is tedious. The <u>same</u> definition leads to the <u>same</u> lemma for topological spaces, abelian groups, modules over a ring, sheaves, ... in fact this lemma is <u>always true</u> (for the "right" notion of fiber product). But we do not want to prove it again and again.

<u>Example 3</u> Let $A \subseteq B_1 \subseteq B_2$ be abelian groups. Then recall Noether's second isomorphism theorem:

$$\frac{B_2/A}{B_1/A} \cong B_2/B_1 \qquad (4.2)$$

Recall the proof. We define $\iota : B_1/A \longrightarrow B_2/A$ by $\iota(\bar{x}) = \bar{x}$ and check that (a) ι is injective and (b) the map $\pi : B_2/A \longrightarrow B_2/B_1$, $\pi(\bar{y}) = \bar{y}$ is surjective and has kernel $\mathrm{Im}(\iota)$ so (c) by Noether's first isomorphism theorem, $\mathrm{Im}(\pi) \cong (B_2/A)/\mathrm{Ker}(\pi)$ and we're done.

<u>Example 4</u> Let X be a topological space and $\mathcal{A} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2$ sheaves of abelian groups on X. Then it is true that

$$\frac{\mathcal{B}_2/\mathcal{A}}{\mathcal{B}_2/\mathcal{A}} \cong \mathcal{B}_2/\mathcal{B}_1$$

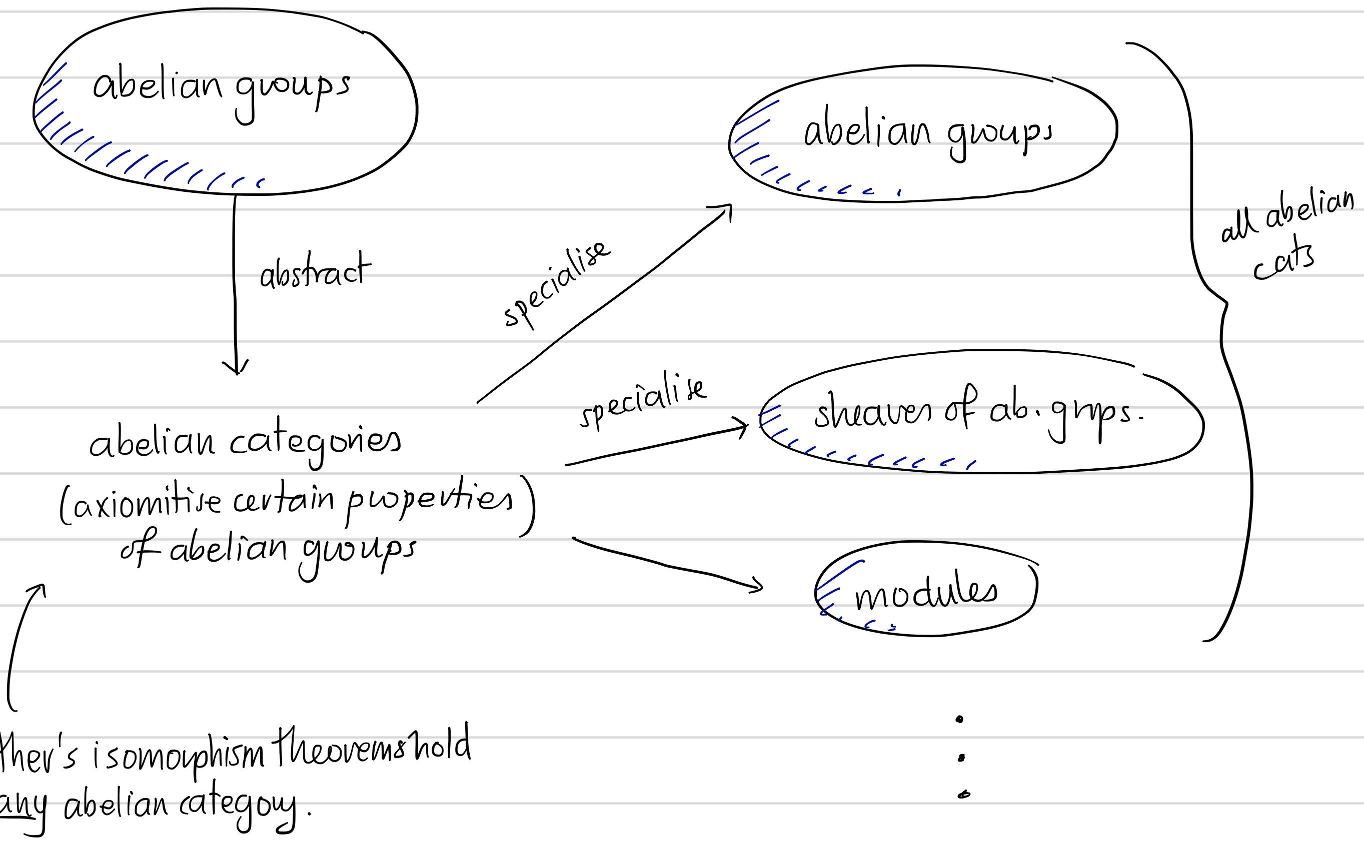
[aside: a <u>pre</u>sheaf $\mathcal{T}$ of abelian groups on X is an abelian group $\mathcal{T}(U)$ for every open $U \subseteq X$ and homomorphisms $\mathcal{T}(U) \to \mathcal{T}(V)$ for all pairs $V \subseteq U$, satisfying an obvious compatibility] but while the proof is "morally" the same as for abelian groups, it is not literally the same, because for sheaves surjectivity (on sections) makes sense but is the "wrong" notion of epimorphism, and

$$\left(\mathcal{B}_2/\mathcal{A}\right)(U) \neq \mathcal{B}_2(U)/\mathcal{A}(U) \qquad (5.1)$$

so the argument does not naively reduce to the case of abelian groups.

<u>The point</u>: there are <u>many</u> theorems about abelian groups (like Noether's isomorphism theorems) which are also true for sheaves of abelian groups, (resp. R-modules and sheaves of modules) and we want a formalism that will allow us to prove these theorems <u>once</u> and "reuse" them.

That formalism is <u>category theory</u>. In outline



Regarding our first example, the relevant abstraction is a category with pullbacks (a pullback is the axiomitised / abstract version of a fiber product)

$$\begin{tikzcd} & \text{topological spaces} \\ \text{category with pullbacks} \arrow[ur, "\text{specialise}"] \arrow[r, "\text{specialise}"'] & \text{abelian groups} \\ & \vdots \end{tikzcd}$$

these are categories with pullbacks.

$V\times_Y(X\times_Z Y) \cong V\times_Z X$ holds here

It's time for definitions. We assume given some set-theoretic foundation in which it makes sense to talk of <u>sets</u> and <u>classes</u> (every set is a class but not vice-versa) such that there is a <u>class of all sets</u>. See A1 (means appendix 1).

<u>Def^N</u> A category $\mathcal{C}$ consists of

(1) A class $ob(\mathcal{C})$ whose elements are called <u>objects</u> of the category

(2) For each pair of objects A, B a set $\mathcal{C}(A,B)$ whose elements are called <u>morphisms</u> from A to B and are written $f: A \longrightarrow B$. (also called <u>arrows</u> from A to B).

(3) For every triple of objects (A,B,C) a function

$$c_{ABC} : \mathcal{C}(B,C)\times\mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$$

called <u>composition</u> and written $g\circ f = c_{ABC}(g,f)$.

(4) For each object A, a morphism $1_A \in \mathscr{C}(A,A)$ called the identity on A.

Satisfying the following axioms:

(1) Associativity For any tuple (A,B,C,D) of objects and morphisms as indicated in the diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(2) Units For any morphism $f : A \longrightarrow B$ we have

$$1_B \circ f = f = f \circ 1_A.$$

Def^n A diagram of morphisms

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ C \arrow[r, "f'"'] & D \end{tikzcd}$$

in a category $\mathscr{C}$ is said to commute if $g \circ f = f' \circ g'$.

Note Often $g \circ f$ is abbreviated to gf.

Above Example 1 / Example 2 suggested that the concept of a fiber-product could be usefully abstracted, while Example 3 / Example 4 suggest we abstract (among possibly other things) the concept of quotient by a subgroup. We do the first abstraction immediately:

Def^N Given a pair of morphisms in a category $\mathcal{C}$

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y \arrow[r, "\theta"'] & Z \end{tikzcd} \tag{8.1}$$

a triple (U, p_X, p_Y) consisting of an object U and morphisms $p_X : U \longrightarrow X$, $p_Y : U \longrightarrow Y$ is called a pullback of (8.1) if

- the diagram $\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$ commutes, i.e. $\pi p_X = \theta p_Y$.

- for any pair of morphisms $\alpha_X : V \longrightarrow X$, $\alpha_Y : V \longrightarrow Y$ with $\pi \alpha_X = \theta \alpha_Y$ there is a unique morphism $\alpha : V \longrightarrow U$ such that $p_X \alpha = \alpha_X$ and $p_Y \alpha = \alpha_Y$.

we call this the "universal property" of the pullback

$$\begin{tikzcd} V \arrow[dr, dashed, "\alpha"'] \arrow[drr, bend left, "\alpha_X"] \arrow[ddr, bend right, "\alpha_Y"'] & & \\ & U \arrow[r, "p_X"'] \arrow[d, "p_Y"] & X \arrow[d, "\pi"] \\ & Y \arrow[r, "\theta"'] & Z \end{tikzcd} \tag{8.2}$$

Lemma 1 A pullback, if it exists, is unique up to unique isomorphism

Proof Next time. $\square$

So we may speak of "the" pullback, properly understood.

Def^N Let **Top** denote the category whose objects are topological spaces, and whose morphisms are continuous maps.

Lemma 2 If (8.1) is a diagram in **Top**, the pullback is the fiber product $(X \times_Z Y, p_X, p_Y)$.

Proof What we mean is that p_X, p_Y have the universal property. Suppose V, α_X, α_Y are given with $\pi \alpha_X = \theta \alpha_Y$, and define

$$\alpha : V \longrightarrow X \times_Z Y$$
$$\alpha(v) = (\alpha_X(v), \alpha_Y(v))$$

The fact that $\pi \alpha_X(v) = \theta \alpha_Y(v)$ means this is well-defined, and it is clearly continuous, and satisfies $p_X \alpha = \alpha_X$, $p_Y \alpha = \alpha_Y$. But is α unique with these properties? Yes, because if α' were another morphism with $p_X \alpha' = \alpha_X$, $p_Y \alpha' = \alpha_Y$ then for all v, by def^N

$$\alpha'(v) = (p_X \alpha'(v), p_Y \alpha'(v))$$
$$= (\alpha_X(v), \alpha_Y(v)) = \alpha(v)$$

since $\alpha'(v) \in X \times_Z Y \subseteq X \times Y$.

Hence $\alpha = \alpha'$! $\square$

This shows that pullbacks give an abstract notion of fiber products. It remains to argue that the <u>lemma</u> about fiber products, which we observed to be true in <u>Top</u> and the category of abelian groups, is the shadow of a theorem about pullbacks:

<u>Theorem</u> Let $\mathcal{C}$ be a category and consider a commutative diagram

means both squares commute separately

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & Z \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

in which II is a pullback. Then I is a pullback if and only if the outer square (X, Z, A, C) is a pullback.

We will prove this next time. But observe that as a special case in <u>Top</u>,

$$V \times_Y (X \times_Z Y) \cong \text{pullback of} \quad \begin{tikzcd} & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

Lemma 2

by the Thm

$$\cong \text{pullback of} \quad \begin{tikzcd} & X \arrow[d] \\ V \arrow[r] & Z \end{tikzcd}$$

Lemma 2

$$\cong V \times_Z X$$

as claimed.

Appendix

References to notes on my website thevisingsea.org are often by acronyms (e.g. FCT).

$\boxed{A1}$ See my (FCT) notes and the introduction to Borceaux "Handbook of categorical algebra"