MAST90068 – Lecture 10

In this lecture we (finally) discuss <u>adjunctions</u>, which have secretly been playing an important role since the beginning of the course.

<u>Def^N</u> Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors. An <u>adjunction</u> with G on the left and F on the right is a family of bijections

$$\theta_{AB} : \mathrm{Hom}_{\mathcal{A}}(GB, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(B, FA), \quad A \in \mathrm{ob}(\mathcal{A}),\ B \in \mathrm{ob}(\mathcal{B})$$

which is natural in A, B by which we mean that for every morphism $f : A \longrightarrow A'$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(GB, A) \arrow[r, "\theta_{AB}"] \arrow[d, "f \circ -"'] & \mathrm{Hom}_{\mathcal{B}}(B, FA) \arrow[d, "F(f) \circ -"] \\
\mathrm{Hom}_{\mathcal{A}}(GB, A') \arrow[r, "\theta_{A'B}"'] & \mathrm{Hom}_{\mathcal{B}}(B, FA')
\end{tikzcd} \qquad (1.1)$$

commutes, and for every morphism $g : B \longrightarrow B'$ in $\mathcal{B}$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(GB, A) \arrow[r, "\theta_{AB}"] & \mathrm{Hom}_{\mathcal{B}}(B, FA) \\
\mathrm{Hom}_{\mathcal{A}}(GB', A) \arrow[r, "\theta_{AB'}"'] \arrow[u, "- \circ Gg"] & \mathrm{Hom}_{\mathcal{B}}(B', FA) \arrow[u, "- \circ g"']
\end{tikzcd} \qquad (1.2)$$

commutes.

If such an adjunction exists we say G is <u>left adjoint</u> to F, that F is <u>right adjoint</u> to G, and write $G \dashv F$. We call (F, G) an <u>adjoint pair</u>.

Note As we will show, there is essentially at most one adjunction (i.e. family of θ's) between a pair F, G and the right (resp. left) adjoint of a functor is essentially unique if it exists.

Examples (1) (Lecture 2, p. 10) There is an adjoint pair

$$\begin{tikzcd} \underline{Set} \arrow[r, bend left, "V"] & \underline{Ab} \arrow[l, bend left, "F"] \end{tikzcd}$$

F = forgetful, V = free ab. grp.

$$\mathrm{Hom}_{\underline{Ab}}(VS, B) \cong \mathrm{Hom}_{\underline{Set}}(S, FB) \quad \text{so} \quad V \dashv F$$

(2) (Lecture 2, Ex 11) There is an adjoint pair

$$\begin{tikzcd} \underline{Mon} \arrow[r, bend left, "V"] & \underline{Rng} \arrow[l, bend left, "F"] \end{tikzcd}$$

F = forgetful, V = free ring on a monoid

$$\mathrm{Hom}_{\underline{Rng}}(VG, R) \cong \mathrm{Hom}_{\underline{Mon}}(G, FR) \quad \text{so} \quad V \dashv F$$

(3) (Lecture 7, p. 2) There is an adjoint pair

$$\begin{tikzcd} \underline{Graph} \arrow[r, bend left, "P"] & \underline{Cat} \arrow[l, bend left, "U"] \end{tikzcd}$$

U = underlying, P = path cat.

$$\mathrm{Hom}_{\underline{Cat}}(PG, \mathcal{C}) \cong \mathrm{Hom}_{\underline{Graph}}(G, U\mathcal{C}) \quad \text{so} \quad P \dashv U$$

$\boxed{\text{Upshot : free is left adjoint to forgetful !}}$

(4) Suppose G is an oriented graph, and $\mathcal{C}$ a category in which diagrams of shape G have a limit. Then by Lecture 9 p.7 we have an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "I"] & {[PG, \mathcal{C}]} \arrow[l, bend left, "\lim"] \end{tikzcd} \qquad I = \text{constant diagram} \quad \lim = \text{limit}$$

$$\mathrm{Hom}_{[PG,\mathcal{C}]}(IC, \alpha) \cong \mathrm{Hom}_{\mathcal{C}}(C, \lim \alpha) \quad \text{so } I \dashv \lim$$

(5) $G, \mathcal{C}$ as above, and every diagram of shape G admitting a <u>colimit</u>.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "I"] & {[PG, \mathcal{C}]} \arrow[l, bend left, "\mathrm{colim}"] \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, \alpha, C) \cong \mathrm{Hom}_{[PG,\mathcal{C}]}(\alpha, IC) \quad \text{so } \mathrm{colim} \dashv I.$$

<u>Theorem</u> Consider functors $F: \mathcal{A} \longrightarrow \mathcal{B}$ and $G: \mathcal{B} \longrightarrow \mathcal{A}$. TFAE

(1) There is an adjunction θ with G on the left and F on the right.

(2) There exist natural transformations $\eta: 1_{\mathcal{B}} \longrightarrow F \circ G$ and $\varepsilon: G \circ F \longrightarrow 1_{\mathcal{A}}$ such that for all $A \in ob(\mathcal{A})$ and $B \in ob(\mathcal{B})$

$$FA \xrightarrow{\eta_{FA}} FGFA \xrightarrow{F(\varepsilon_A)} FA \;=\; FA \xrightarrow{id} FA$$
$$GB \xrightarrow{G(\eta_B)} GFGB \xrightarrow{\varepsilon_{GB}} GB \;=\; GB \xrightarrow{id} GB \qquad (3.1)$$

(3) There exists a natural transformation $\eta : 1_{\mathcal{B}} \longrightarrow F \circ G$ such that for every $B \in ob(\mathcal{B})$ the morphism $\eta_B : B \longrightarrow FGB$ is <u>universal</u> among morphisms from B to objects of the form FA. That is, for any morphism $b : B \longrightarrow FA$ there is a unique morphism $a : GB \longrightarrow A$ such that

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[dr, "b"'] & FGB \arrow[d, dashed, "Fa"] \\ & FA \end{tikzcd} \qquad (4.1)$$

commutes.

(4) There exists a natural transformation $\varepsilon : G \circ F \longrightarrow 1_{\mathcal{A}}$ such that for every $A \in ob(\mathcal{A})$ the morphism $\varepsilon_A : GFA \longrightarrow A$ is <u>universal</u> among morphisms from object of the form GB to A. That is, for any morphism $a : GB \longrightarrow A$ there is a unique morphism $b : B \longrightarrow FA$ s.t.

$$\begin{tikzcd} GFA \arrow[r, "\varepsilon_A"] & A \\ & GB \arrow[u, "a"'] \arrow[ul, dashed, "Gb"] \end{tikzcd} \qquad (4.2)$$

commutes.

<u>Proof</u> We prove $(1) \Leftrightarrow (3)$ and leave $(2) \Leftrightarrow (1) \Leftrightarrow (4)$ as an exercise.
First, $(1) \Rightarrow (3)$. Suppose we have natural bijections

$$\theta_{AB} : Hom_{\mathcal{A}}(GB, A) \xrightarrow{\cong} Hom_{\mathcal{B}}(B, FA),$$

and observe that taking $A = GB$ gives

$$\theta_{GB,B} : \mathrm{Hom}_{\mathcal{A}}(GB, GB) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(B, FGB),$$

and we define

$$\eta_B := \theta_{GB,B}(1_{GB}) : B \longrightarrow FGB.$$

We need to check this is (a) natural and (b) universal. For naturality, suppose $f : B \to B'$ is given. By naturality of θ, both squares in

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(GB, GB) \arrow[r, "\theta_{GB,B}"] \arrow[d, "Gf\circ -"'] & \mathrm{Hom}_{\mathcal{B}}(B, FGB) \arrow[d, "FG(f)\circ -"] \\
\mathrm{Hom}_{\mathcal{A}}(GB, GB') \arrow[r, "\theta_{GB',B}"'] & \mathrm{Hom}_{\mathcal{B}}(B, FGB') \\
\mathrm{Hom}_{\mathcal{A}}(GB', GB') \arrow[r, "\theta_{GB',B'}"'] \arrow[u, "-\circ Gf"] & \mathrm{Hom}_{\mathcal{B}}(B', FGB') \arrow[u, "-\circ f"']
\end{tikzcd} \tag{5.1}$$

commute. The top one says

$$FG(f) \circ \theta_{GB,B}(1_{GB}) = \theta_{GB',B}(Gf \circ 1_B)$$

$$\text{i.e.} \quad FG(f) \circ \eta_B = \theta_{GB',B}(Gf)$$

while the bottom one says

$$\theta_{GB',B}(1_{GB} \circ Gf) = \theta_{GB',B'}(1_{GB'}) \circ f$$

$$\text{i.e. } \theta_{GB',B}(Gf) = \eta_{B'} \circ f$$

We conclude $FG(f) \circ \eta_B = \eta_{B'} \circ f$ so η is a natural transformation $1_{\mathcal{B}} \to F \circ G$. It remains to prove the universal property, but we will derive this easily from the following observation: given any $a : GB \to A$ (that is, something to which $\theta_{A,B}$ may be applied) we may apply naturality of θ to deduce commutativity of

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(GB, GB) \arrow[r, "\theta_{GB,B}"] \arrow[d, "a \circ -"'] & \mathrm{Hom}_{\mathcal{B}}(B, FGB) \arrow[d, "F(a) \circ -"] \\
\mathrm{Hom}_{\mathcal{A}}(GB, A) \arrow[r, "\theta_{GB,A}"'] & \mathrm{Hom}_{\mathcal{B}}(B, FA).
\end{tikzcd} \qquad (6.1)$$

But this says in particular

$$\begin{aligned}
\theta_{GB,A}(a) &= \theta_{GB,A}(a \circ 1_{GB}) \\
&= F(a) \circ \theta_{GB,B}(1_{GB}) \qquad (6.2) \\
&= F(a) \circ \eta_B .
\end{aligned}$$

Thus $\underline{\theta \text{ is determined by } \eta}$, just as $\underline{\eta \text{ is determined by } \theta}$. To see why this implies (3), let $b : B \to FA$ be given. Considering (6.1) and the fact that θ is a bijection, there must be a unique $a : GB \to A$ with $\theta_{GB,A}(a) = b$. But this is the same (by (6.2)) as to say there is a unique a with $b = F(a) \circ \eta_B$, which is what we wanted to show. Hence (1) $\Rightarrow$ (3).

For (3) $\Rightarrow$ (1) suppose we have $\eta : 1_{\mathcal{B}} \to F \circ G$ with the universal property and define the function

$$\theta_{A,B} : \mathrm{Hom}_{\mathcal{A}}(GB, A) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(B, FA) \tag{7.1}$$

$$\theta_{A,B}(a) = F(a) \circ \eta_B$$

$$\text{i.e.} \quad B \xrightarrow{\eta_B} FGB \xrightarrow{Fa} FA$$

The universal property of η says precisely that $\theta_{A,B}$ is a bijection. It remains to check naturality of θ. Given $f : A \to A'$ and $g : B \to B'$ we have to check commutativity of

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(GB, A) \arrow[r, "\theta_{A,B}"] \arrow[d, "f \circ -"'] & \mathrm{Hom}_{\mathcal{B}}(B, FA) \arrow[d, "Ff \circ -"] \\
\mathrm{Hom}_{\mathcal{A}}(GB, A') \arrow[r, "\theta_{A'B}"'] & \mathrm{Hom}_{\mathcal{B}}(B, FA')
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\checkmark"] & F(a)\eta_B \arrow[d, mapsto] \\
fa \arrow[r, mapsto] & F(f) \circ (F(a)\eta_B) = F(fa)\eta_B
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(GB, A) \arrow[r, "\theta_{AB}"] & \mathrm{Hom}_{\mathcal{B}}(B, FA) \\
\mathrm{Hom}_{\mathcal{A}}(GB', A) \arrow[r, "\theta_{AB'}"'] \arrow[u, "- \circ Gg"] & \mathrm{Hom}_{\mathcal{B}}(B', FA) \arrow[u, "- \circ g"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a \circ G(g) \arrow[r, mapsto] & F(a \circ Gg)\eta_B = F(a)FG(g)\eta_B \overset{\eta \text{ is natural}}{=} F(a)\eta_{B'}g \\
a \arrow[u, mapsto] \arrow[ur, phantom, "\checkmark"] \arrow[r, mapsto, bend right] & F(a)\eta_{B'} \arrow[u, mapsto]
\end{tikzcd}$$

So this shows (3) $\Rightarrow$ (1). $\square$

Ex 1 Complete the proof of the theorem by proving (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (4).

Def^n In an adjunction the natural transformation $\eta : 1_{\mathcal{B}} \longrightarrow F \circ G$ is called the unit and $\varepsilon : G \circ F \longrightarrow 1_{\mathcal{A}}$ the counit.

Ex 2 Write down the unit and counit transformations in the examples (1)–(5) on p. 2, 3. Here you will use the explicit θ's defined in earlier lectures.

Ex 3 Prove that if a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ has a right (resp. left) adjoint, it is unique up to isomorphism (and in fact up to unique isomorphism if we add a natural constraint involving either the unit or counit. What is the right constraint?).

Ex 4 Given a pair of functors F, G as above, to what extent is an adjunction θ between them unique (if it exists)?