MAST90068 – Lecture 11

Now we know category theory and it is time to turn to homological algebra (our main application), where the categories of interest are categories of modules over rings, and more generally abelian categories. The subject is primarily concerned with complexes and their cohomology, and we will take the following as our main motivations:

1. The theory of homology $H_i(X)$ and cohomology $H^i(X)$ of a topological space X, defined as cohomology of certain complexes associated functorially to X.

2. The Auslander-Buchsbaum-Serre theorem, which characterises regularity of local noetherian rings (i.e. smoothness of points on algebraic varieties) as finiteness of global dimension (an invariant belonging to homological algebra), and Hilbert's syzygy theorem which shows the global dimension of $k[x_1,\dots,x_n]$ for k a field is equal to n.

These examples show the fundamental role homological algebra plays in algebraic topology resp. algebraic geometry.

Conventions All rings are associative and unital, ring morphisms preserve units.

Recall if R is a ring, a left R-module is an abelian group M with ring morphism $\gamma_M : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$, where $\mathrm{End}_{\mathbb{Z}}(M)$ denotes morphisms of abelian groups $M \longrightarrow M$. We usually write $r \cdot m$ or rm for $\gamma_M(r)(m)$. A morphism of left R-modules $\phi : M \longrightarrow N$ is a morphism of abelian groups s.t. for all $r \in R$, $m \in M$ we have $\phi(r \cdot m) = r \cdot \phi(m)$. A right R-module is a left R^{op}-module, where R^{op} is the abelian group R with multiplication $r * s = sr$.

The category of left R-modules is denoted R-Mod and the category of right R-modules is denoted Mod-R. We write for left R-modules M, N

$$\mathrm{Hom}_R(M,N) := \mathrm{Hom}_{R\text{-Mod}}(M,N)$$

and similarly for right modules.

<u>Note</u> $\mathbb{Z}$-Mod $=$ <u>Ab</u>, Mod-$R = R^{op}$-Mod. (so we need only state some things once)

<u>Note</u> We will assume basic familiarity with the theory of modules, for a refresher see e.g. Hilton & Stammbach.

<u>Ex 1</u> In R-Mod, mono = injective and epi = surjective.

<u>Def^N</u> In a category $\mathcal{C}$ a <u>zero object</u> is an object 0 which is both an initial and terminal object, i.e. for all $C \in ob(\mathcal{C})$ we have both $\mathrm{Hom}_{\mathcal{C}}(0, C)$ and $\mathrm{Hom}_{\mathcal{C}}(C, 0)$ singletons. We denote these morphisms also by 0 or $0_C : 0 \to C$, $0_C : C \to 0$ and for any $A, B \in ob(\mathcal{C})$ write 0_{AB} for the morphism $A \longrightarrow 0 \longrightarrow B$.

<u>Def^N</u> Recall that for a morphism of R-modules $\psi : M \longrightarrow N$,

$$\begin{aligned} \mathrm{Ker}(\psi) &= \{ m \in M \mid \psi(m) = 0 \} \subseteq M \\ \mathrm{Im}(\psi) &= \{ \psi(m) \mid m \in M \} \subseteq N \\ \mathrm{Coker}(\psi) &= N / \mathrm{Im}(\psi). \end{aligned} \tag{2.1}$$

are all R-modules.

Def^N Let $\mathscr{C}$ be a category with zero object, and $f: C \to C'$ a morphism.
The <u>kernel</u> of f (if it exists) is the pullback of

$$\begin{tikzcd} & 0 \arrow[d] \\ C \arrow[r, "f"'] & C' \end{tikzcd} \tag{3.1}$$

The <u>cokernel</u> of f (if it exists) is the pushout of

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & C' \\ 0 & \end{tikzcd} \tag{3.2}$$

That is, the kernel is a pair $(\mathrm{Ker}(f), u)$ consisting of an object $\mathrm{Ker}(f)$ and morphism $u: \mathrm{Ker}(f) \to C$ s.t. $f \circ u = 0$ and for any other $v: X \to C$ with $f \circ v = 0$ there is a unique $h: X \to \mathrm{Ker}(f)$ making the diagram

$$\begin{tikzcd} X \arrow[dr, "v"] \arrow[d, dashed, "h"'] & & \\ \mathrm{Ker} f \arrow[r, "u"'] & C \arrow[r, "f"] & C' \end{tikzcd} \tag{3.3}$$

commute. If kernels (resp. cokernels) exist for all morphisms we say $\mathscr{C}$ <u>has kernels</u> (resp. <u>has cokernels</u>).

<u>Ex 2</u> Write down a similar explicit form of the universal property of the cokernel.

<u>Ex 3</u> Check that in R-mod, for any $\psi: M \to N$, the inclusion $\mathrm{Ker}(\psi) \to M$ is the kernel and the quotient $N \to \mathrm{Coker}(\psi)$ is the cokernel.

Def Let $\mathscr{C}$ be a category. A <u>subobject</u> of an object C is a mono $u: S \longrightarrow C$. We say u <u>precedes</u> another subobject $u': S' \longrightarrow C$ if there is a <u>factorisation</u> of u through u', i.e. there exists $w: S \longrightarrow S'$ with $u' \circ w = u$, and we write $u \leq u'$ or by abuse of notation $S \leq S'$.

Def Let $\mathscr{C}$ be a category, and $f: C \to C'$ a morphism. The <u>image</u> of f is the smallest subobject of C' through which f factors, that is, it is a subobject $u: I \to C'$ through which f factors (note this factorisation $f': C \to I$ with $f = u \circ f'$ is unique) which precedes any other subobject of C' with this property. If the image of f exists it is unique up to unique isomorphism and we denote it $Im(f)$. If every morphism in $\mathscr{C}$ has an image we say it <u>has images</u>.

<u>Ex 4</u> Suppose $\mathscr{C}$ has equalisers, then the factorisation f' of f through its image is always epi.

<u>Def^N</u> We say a category $\mathscr{C}$ is <u>balanced</u> if epi + mono $\Rightarrow$ iso.

<u>Ex 5</u> If $\mathscr{C}$ is balanced and $f: C \to C'$ has a factorisation $C \xrightarrow{g} I \xrightarrow{h} C'$ with h mono and g epi then h is the image of f (assume f <u>has</u> an image).

<u>Ex 6</u> In R-Mod, the inclusion $Im(\varphi) \hookrightarrow N$ is the image of $\varphi: M \to N$.

<u>Def^N</u> A <u>complex</u> of R-modules is a collection of R-modules $\{C^n\}_{n\in\mathbb{Z}}$ and R-linear maps $\{\partial^n: C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$ such that $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{Z}$, i.e.

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \cdots \qquad \partial \circ \partial = 0.$$

We tend to write (C, ∂) or just C to stand for this data, and call ∂ the <u>differential</u> of the complex.

<u>Note</u> This is called <u>cohomological indexing</u>. An equivalent def^{n} but with <u>homological indexing</u> is $\{C_n\}_{n\in\mathbb{Z}}$ with $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ s.t. $\partial_{n-1}\circ\partial_n = 0$ for all $n\in\mathbb{Z}$,

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

Sometimes one says "cochain complex" for a cohomologically indexed complex and "chain complex" for a homologically indexed complex.

<u>Note</u> Equivalently, a complex is a $\mathbb{Z}$-graded R-module C with a degree $+1$ R-linear map $\partial : C \to C$ satisfying $\partial^2 = 0$.

<u>Def^{N}</u> Let C, D be complexes of R-modules. A <u>morphism</u> $\alpha : C \to C'$ is a family of morphisms of R-modules $\{\alpha^n : C^n \to D^n\}_{n\in\mathbb{Z}}$ such that for all $n\in\mathbb{Z}$ the diagram

$$\begin{tikzcd}
C^n \arrow[r, "\partial^n"] \arrow[d, "\alpha^n"'] & C^{n+1} \arrow[d, "\alpha^{n+1}"] \\
D^n \arrow[r, "\partial^n"'] & D^{n+1}
\end{tikzcd}$$

commutes. With composition defined for $\beta : C' \to C''$ by $(\beta\circ\alpha)^n = \beta^n \circ \alpha^n$ and $\mathrm{id}_C : C \to C$ given by $(\mathrm{id}_C)^n = \mathrm{id}_{C^n}$ this defines the <u>category of complexes</u> of R-modules $Ch(R)$.

<u>Def^{n}</u> (Cohomology) Let (C,∂) be a complex of R-modules, then

$$Z^n(C) := \mathrm{Ker}(\partial^n : C^n \to C^{n+1}) \subseteq C^n \qquad \text{called } \underline{\text{cocycles}}$$
$$B^n(C) := \mathrm{Im}(\partial^{n-1} : C^{n-1} \to C^n) \subseteq C^n \qquad \text{called } \underline{\text{coboundaries}}$$

Since $\partial^n \circ \partial^{n-1} = 0$ we have $B^n(C) \subseteq Z^n(C)$ and the <u>nth cohomology</u> of C is the quotient

$$H^n(C) := Z^n(C)/B^n(C). \tag{6.1}$$

<u>Lemma</u> For each $n \in \mathbb{Z}$ there is a functor $H^n(-) : Ch(R) \longrightarrow R\text{-Mod}$, defined on objects by (6.1).

<u>Proof</u> Let $\alpha : C \to D$ be a morphism. Then there is a commutative diagram

$$\begin{tikzcd} Z^n(C) \arrow[r, hook] \arrow[d, dashed, "Z^n(\alpha)"'] & C^n \arrow[r, "\partial^n"] \arrow[d, "\alpha^n"'] & C^{n+1} \arrow[d, "\alpha^{n+1}"] \\ Z^n(D) \arrow[r, hook] & D^n \arrow[r, "\partial^n"'] & D^{n+1} \end{tikzcd} \tag{6.2}$$

Since $\partial^n \alpha^n(x) = \alpha^{n+1}\partial^n(x) = 0$ for $x \in Z^n(C)$, α^n restricts to a map $Z^n(\alpha) : Z^n C \longrightarrow Z^n D$ making the diagram commute. Moreover $\alpha^n \partial^{n-1}(y) = \partial^{n-1}\alpha^{n-1}(y)$ so α^n also restricts to a map $B^n(\alpha) : B^n C \longrightarrow B^n D$, and the left hand square of

$$\begin{tikzcd} B^n C \arrow[r, hook, "inc"] \arrow[d, "B^n\alpha"'] & Z^n C \arrow[r, two heads] \arrow[d, "Z^n\alpha"] & H^n C \arrow[d, dashed, "H^n\alpha"] \\ B^n D \arrow[r, hook] & Z^n D \arrow[r, two heads] & H^n D \end{tikzcd} \tag{6.3}$$

inc means "inclusion"

commutes, so there is a unique morphism $H^n\alpha$ on the quotients making the right hand diagram commute. Concretely, for $x \in Z^n(C)$,

$$H^n(\alpha)(\bar{x}) = \overline{\alpha^n(x)}$$

It is therefore clear $H^n(-)$ is a functor since

$$H^n(1_c)(\bar{x}) = \overline{1_{C^n}(x)} = \bar{x} \quad \therefore \quad H^n(1_c) = 1_{H^nC}$$

$$\begin{aligned} H^n(\beta\circ\alpha)(\bar{x}) &= \overline{(\beta\circ\alpha)^n(x)} \\ &= \overline{\beta^n(\alpha^n(x))} \\ &= H^n(\beta)\left(\overline{\alpha^n(x)}\right) \\ &= H^n(\beta)\left(H^n(\alpha)(\bar{x})\right) \\ &= \left(H^n(\beta)\circ H^n(\alpha)\right)(\bar{x}). \end{aligned}$$

□

__Examples__ (1) Let $R = k[\varepsilon]/\varepsilon^2 = k\cdot 1 \oplus k\cdot\varepsilon$ for some field k, and let $(C,\partial)\in Ch(R)$ be $C^n = R$ for all $n\leq 0$, and all nonzero ∂^n be multiplication by ε

$$\cdots \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} \underset{0}{R} \longrightarrow 0 \longrightarrow 0 \cdots$$

Then for any $n<0$

$$\begin{aligned} H^n(C) &= \frac{\operatorname{Ker}(R \xrightarrow{\varepsilon} R)}{\operatorname{Im}(R \xrightarrow{\varepsilon} R)} \\ &= k\varepsilon / k\varepsilon = 0 \end{aligned}$$

$$H^0(C) = \frac{\operatorname{Ker}(R \longrightarrow 0)}{\operatorname{Im}(R \xrightarrow{\varepsilon} R)} = R/_{k\varepsilon} = k\cdot 1$$

Clearly $H^n(C) = 0$ for all $n>0$, so $H^n(C) = \begin{cases} 0 & n\neq 0 \\ k & n=0. \end{cases}$

(2) Let $R = k[x,y]$, k a field, and consider the complex

$$C: \quad 0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^{\oplus 2} \xrightarrow{(x\ y)} \underset{0}{R} \longrightarrow 0 \qquad (8.1)$$

with degree zero as marked. Then

$$H^0(C) = \frac{\operatorname{Ker}(R \to 0)}{\operatorname{Im}((x,y))} = R/(x,y) \cong k \quad (\text{with } x, y \text{ acting as zero})$$

$$H^{-1}(C) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in R^{\oplus 2} \,\middle|\, xf + yg = 0 \right\} \Big/ \left\{ \begin{pmatrix} yh \\ -xh \end{pmatrix} \,\middle|\, h \in R \right\}$$

From $xf = -yg$ we deduce $g = -xh$ for some h, but then $xyh = xf$ implies $f = yh$ so $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} yh \\ -xh \end{pmatrix}$ and hence $H^{-1}(C) = 0$.

$$H^{-2}(C) = \operatorname{Ker}\begin{pmatrix} y \\ -x \end{pmatrix} = 0.$$

<u>Note</u> The complexes C in (1), (2) are examples of <u>projective</u> resolutions of k. By the Auslander-Buchsbaum-Serre theorem <u>every</u> projective resolution of k over $R = k[\varepsilon]/\varepsilon^2$ (that is, a complex nonzero only in degrees $n \leq 0$ with each degree a projective R-module and only cohomology k in deg. 0) must be <u>infinite</u>. By Hilbert's syzygy theorem any projective resolution of k over $k[x,y]$ which is longer than (8.1) must be "trivial" past the second differential (we will see details later).

because R is not a regular ring

Ex 7 Let $\mathscr{C}$ be a category with zero, which has kernels, cokernels, images and equalisers. This exercise will guide you through trying to define a cohomology functor $H^n(-) : Ch(\mathscr{C}) \longrightarrow \mathscr{C}$, as we did for $\mathscr{C} = R\text{-Mod}$.

(a) Define (cochain) complexes $\{C^n, \partial^n : C^n \to C^{n+1}\}_{n \in \mathbb{Z}}$ in $\mathscr{C}$ in the obvious way, as well as their morphisms, and check these form a category $Ch(\mathscr{C})$.

(b) Given a commutative diagram in $\mathscr{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

show that

(i) there is a unique $w : \operatorname{Ker} f \to \operatorname{Ker} g$ making

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r] \arrow[d, dashed, "w"'] & A \arrow[d, "u"] \\ \operatorname{Ker} g \arrow[r] & C \end{tikzcd}$$

commute.

(ii) there is a unique $w : \operatorname{Coker} f \to \operatorname{Coker} g$ making

$$\begin{tikzcd} B \arrow[r] \arrow[d, "u"'] & \operatorname{Coker} f \arrow[d, dashed, "w"] \\ C \arrow[r] & \operatorname{Coker} g \end{tikzcd}$$

commute.

(iii) by Ex 4 the factorisations

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f'"'] & & B \\
& \mathrm{Im} f \arrow[ur, "i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "g"] \arrow[dr, "g'"'] & & D \\
& \mathrm{Im} g \arrow[ur, "j"'] &
\end{tikzcd}
$$

have both f', g' epi. Consider the diagram

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f'"] \arrow[ddd, "u"'] & & B \arrow[ddd, "v"] \\
& \mathrm{Im} f \arrow[ur, "i"'] \arrow[d, dotted] \arrow[dl, phantom, "\mathrm{I}"] \arrow[dr, phantom, "\mathrm{II}"] & \\
& \mathrm{Im} g \arrow[dr, "j"] & \\
C \arrow[rr, "g"'] \arrow[ur, "g'"] & & D.
\end{tikzcd}
$$

Is it always true that there is $\mathrm{Im} f \longrightarrow \mathrm{Im} g$ making the two implicit squares II commute? Observe that without this we cannot produce an analogue of (6.3) and thus cannot define cohomology of complexes in $\mathcal{C}$.