MAST90068 – Lecture 12

In this lecture we define singular homology and cohomology of topological spaces. Recall the simplex category Δ, the category of simplicial sets

$$\underline{SSet} = [\Delta^{op}, \underline{Set}]$$

and the functor $S : \underline{Top} \longrightarrow \underline{SSet}$.

<u>Def^N</u> Let $\mathscr{C}$ be a category. Then $[\Delta^{op}, \mathscr{C}]$ is the category of simplicial objects in $\mathscr{C}$ (e.g. simplicial abelian groups, or R-modules).

<u>Theorem</u> A simplicial object in $\mathscr{C}$ is a family $\{C_q\}_{q\geq 0}$ of objects of $\mathscr{C}$ together with two families of morphisms

$$d_i : C_q \longrightarrow C_{q-1} \qquad s_i : C_q \longrightarrow C_{q+1} \qquad i = 0, \dots, q$$

with $q > 0$ in the case of d_i, which satisfy

$$d_i d_j = d_{j-1} d_i \qquad i<j \qquad (1.1)$$
$$s_i s_j = s_{j+1} s_i \qquad i \leq j \qquad (1.2)$$
$$d_i s_j = s_{j-1} d_i \qquad i<j \qquad (1.3)$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i<j \\ 1 & i=j, \text{ or } i=j+1 \\ s_j d_{i-1} & i>j+1 \end{cases} \qquad (1.4)$$

And a morphism $f : C \to C'$ of such simplicial objects is a family of morphisms $\{f_q : C_q \to C'_q\}_{q\geq 0}$ s.t. $d_i f_q = f_{q-1} d_i$ and $s_i f_q = f_{q+1} s_i$ for all q and $0 \leq i \leq q$.

<u>Proof</u> Given a simplicial object $F:\Delta^{op}\to\mathcal{C}$ set $C_q = F([q])$ and

$$d_i = F(\varepsilon^i : [q-1]\to[q]) \quad \text{(in } \Delta\text{)}$$
$$s_i = F(\eta^i : [q+1]\to[q]).$$

The relations (1.1)—(1.4) follow from the relations for ε, η checked in Ex 2 of Lecture 4. A morphism $F\to F'$ is by def^N a family $f_q : C_q \to C_q'$ making the appropriate diagrams commute, including the required identities with d_i, s_i.

Conversely, given the data $\{C_q, s_i, d_i\}_{q\geq 0, 0\leq i\leq q}$ satisfying (1.1)-(1.4) we can define a simplicial object $F:\Delta^{op}\to\mathcal{C}$ by $F([q]) := C_q$ on objects and on $\mu:[q]\to[p]$ using the unique presentation

$$\mu = \varepsilon^{i_1}\cdots\varepsilon^{i_s}\eta^{j_1}\cdots\eta^{j_t} \qquad i_1>\cdots>i_s,\ j_1<\cdots<j_t$$

from the Theorem of Lecture 5, by defining

$$F(\mu) := s_{j_t}\cdots s_{j_1} d_{i_s}\cdots d_{i_1}.$$

To see F is a functor say $\theta:[p]\to[r]$ with $\theta = \varepsilon^{\bar{i}_1}\cdots\varepsilon^{\bar{i}_{\bar{s}}}\eta^{\bar{j}_1}\cdots\eta^{\bar{j}_{\bar{t}}}$ again with $\bar{i}_1>\cdots>\bar{i}_{\bar{s}}$, $\bar{j}_1<\cdots<\bar{j}_{\bar{t}}$. Then $\theta\mu$ may be put into the "ε's after η's" form using the commutation rules of Ex 2 of Lecture 4, and since by hypothesis the same rules apply to s_i, d_i we see $F(\theta\mu) = F(\theta)F(\mu)$, as claimed. □

<u>Lemma</u> Let C be a simplicial abelian group (C_q, s_i, d_i as above). Then with

$$\partial_q : C_q \longrightarrow C_{q-1}, \quad \partial := \sum_{i=0}^{q} (-1)^i d_i$$

we have $\partial_{q-1} \circ \partial_q = 0$ for all q, i.e. (C, ∂) is a chain complex.

<u>Proof</u> We have

$$\begin{aligned}
\partial\partial &= \left(\textstyle\sum_i (-1)^i d_i\right)\left(\textstyle\sum_j (-1)^j d_j\right) \\
&= \textstyle\sum_{i,j} (-1)^{i+j} d_i d_j \\
&= \textstyle\sum_{i<j} (-1)^{i+j} d_i d_j + \sum_{i\geq j} (-1)^{i+j} d_i d_j \\
&= \textstyle\sum_{i<j} (-1)^{i+j} d_{j-1} d_i + \sum_{i\geq j} (-1)^{i+j} d_i d_j \\
&= \textstyle\sum_{i<k+1} (-1)^{i+k+1} d_k d_i + \sum_{i\geq j} (-1)^{i+j} d_i d_j \\
&= \textstyle\sum_{i\leq k} (-1)^{i+k+1} d_k d_i + \sum_{i\geq j} (-1)^{i+j} d_i d_j \\
&= 0. \quad \square
\end{aligned}$$

<u>Ex 1</u> The constuction $C \mapsto (C, \partial)$ extends to a functor $\mathrm{comp} : [\Delta^{op}, \underline{Ab}] \longrightarrow Ch(\mathbb{Z})$, i.e. $\mathrm{comp}(\{C_q, s_i, d_i\}) = (C, \partial = \Sigma_i (-1)^i d_i)$ and from $f : C \to C'$ we obtain a morphism of complexes with $f_q : C_q \to C'_q$ in degree q.

Let $V : \underline{Set} \longrightarrow \underline{Ab}$ be the free abelian group functor from Lecture 2. Postcomposition with V defines a functor (see p. ① Lecture 6)

$$V \circ - \; : \; \underline{SSet} = [\Delta^{op}, \underline{Set}] \longrightarrow [\Delta^{op}, \underline{Ab}].$$

Def^N Let $Ch^\bullet(R)$, $Ch_\bullet(R)$ denote respectively the categories of cochain complexes of R-modules and chain complexes. Last lecture we defined functors $H^n(-) : Ch^\bullet(R) \longrightarrow R\text{-Mod}$ and similarly one defines $H_n(-) : Ch_\bullet(R) \longrightarrow R\text{-Mod}$, where the nth homology of a chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

is defined by

$$H_n(C) := \frac{Z_n(C)}{B_n(C)} = \frac{Ker(\partial_n)}{Im(\partial_{n+1})}.$$

Def^N (Homology groups of a topological space) We define a functor $H_n(-,\mathbb{Z})$ to be the composite $H_n(-,\mathbb{Z}) : \underline{Top} \longrightarrow \underline{Ab}$ given by

$$\underline{Top} \xrightarrow{S} \underline{SSet} = [\Delta^{op}, \underline{Set}] \xrightarrow{V \circ -} [\Delta^{op}, \underline{Ab}] \xrightarrow{comp} Ch_\bullet(\mathbb{Z}) \xrightarrow{H_n} \underline{Ab}.$$

That is, for a topological space X, $H_n(X,\mathbb{Z})$ is the homology of the cochain complex with differential

$$\underbrace{V\Big(\sum_{i=0}^{q}(-1)^i d_i\Big)}_{=\,\partial_n} : \underbrace{V\big((SX)([n])\big)}_{=\,C_n} \longrightarrow \underbrace{V\big((SX)([n-1])\big)}_{=\,C_{n-1}}$$

Now by def^N

$$C_n = V((SX)([n])) = V(\mathrm{Hom}_{\underline{Top}}(\Delta^n, X)) = \bigoplus_{\substack{x:\Delta^n \to X \\ \text{in } \underline{Top}}} \mathbb{Z}x \qquad (4.1)$$

We call such x singular n-simplices, so C_n is the set of formal linear combinations of n-simplices, called n-chains. By def^N, $V(d_i): C_n \to C_{n-1}$ sends a generator $x: \Delta^n \to X$ to $d_i(x) = (SX)(\varepsilon^i)(x) = x \circ \Delta^{\varepsilon^i}$ where

$$\Delta^{\varepsilon^i}: \Delta^{n-1} \to \Delta^n \text{ is } (a_0, \dots, a_n) \mapsto (a_0, \dots, a_{i-1}, 0, a_i, \dots, a_n) \qquad (4.2)$$

↑ this is the inclusion of the ith face of Δ^n, d_i is called the ith face map of the simplicial object SX.

Thus $\partial_n : C_n \to C_{n-1}$ is more concretely

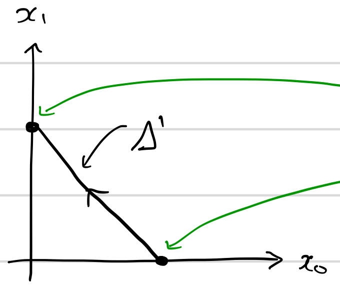
$$\partial_n : \bigoplus_{x:\Delta^n \to X} \mathbb{Z}x \longrightarrow \bigoplus_{y:\Delta^{n-1} \to X} \mathbb{Z}y \qquad (4.3)$$

$$\partial_n(x) = \sum_{i=0}^{n} (-1)^i x \circ \Delta^{\varepsilon^i}$$

Def^N A singular n-chain (i.e. element of $\bigoplus_{x:\Delta^n \to X} \mathbb{Z}x$) in $\mathrm{Ker}(\partial_n)$ is called a cycle while a linear combination in $\mathrm{Im}(\partial_{n+1})$ is called a boundary. So $H_n(X, \mathbb{Z})$ is the group of cycles mod boundaries.

To see what this means, consider the following example:

Example Consider the inclusion $\iota : \Delta^1 \hookrightarrow \mathbb{R}^2$, as in

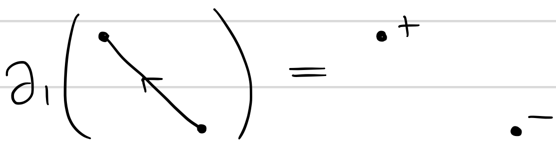


$\Delta^{\varepsilon^i} : \Delta^0 \longrightarrow \Delta^1$

$\Delta^{\varepsilon^0}(1) = (0,1)$

$\Delta^{\varepsilon^1}(1) = (1,0)$

Then $\partial_1(\iota) = \iota \circ \Delta^{\varepsilon^0} - \iota \circ \Delta^{\varepsilon^1}$ as a linear combination of 0-simplices in $\mathbb{R}^2$. But $\Delta^0 = \{1\} \subseteq \mathbb{R}^1$ is just a point, so we may identify 0-simplices with points in which case $\partial_1(\iota) = (0,1) - (1,0)$ (not as vectors, this is a formal linear sum of points in $\mathbb{R}^2$). Following standard convention we view the signs as providing an orientation in the indicated direction, so in summary



Lemma If X is path connected, $H_0(X, \mathbb{Z}) \cong \mathbb{Z}$.

Proof By def^n, $H_0(X,\mathbb{Z}) = \bigoplus_{x : \Delta^0 \to X} \mathbb{Z}x \Big/ \operatorname{Im}(\partial_1)$. Choose a point $x_0 \in X$ and define maps

$$H_0(X,\mathbb{Z}) \xrightarrow{f} \mathbb{Z} \qquad f\left(\textstyle\sum_{x\in X} a_x x\right) = \textstyle\sum_{x \in X} a_x$$

$$\mathbb{Z} \xrightarrow{g} H_0(X,\mathbb{Z}) \qquad g(a) = a x_0 .$$

Observe f is well-defined since for a path $y : \Delta^1 \longrightarrow X$, we have

$$f(\partial_1(y)) = f(y \circ \Delta^{\varepsilon^0} - y \circ \Delta^{\varepsilon^1}) = 1 - 1 = 0.$$

Clearly $f \circ g = id$, and given $\sum_{x \in X} a_x x$ write it as $\sum_{i=1}^{k} a_i x_i$ and let γ_i be any path in X from x_i to x_0, viewed as a continuous map $\gamma_i : \Delta^1 \to X$ (to be clear, $\gamma_i(1,0) = x_i$, $\gamma_i(0,1) = x_0$). Then $\sum_{i=1}^{k} a_i \gamma_i$ is a 1-chain in X, with

$$\begin{aligned} \partial_1\left(\textstyle\sum_i \gamma_i\right) &= \textstyle\sum_i \partial_1(\gamma_i) \\ &= \textstyle\sum_i (a_i x_i - a_i x_0) \\ &= \textstyle\sum_i a_i x_i - \left(\sum_i a_i\right) x_0 \end{aligned}$$

This shows that in $H_0(X, \mathbb{Z})$, $\sum_i a_i x_i \equiv (\sum_i a_i) x_0$ so g is surjective and hence a bijection. □