MAST90068 – Lecture 13

Last lecture we defined the singular homology $H_n X := H_n(X, \mathbb{Z})$ of a topological space X and computed $H_0 X \cong \mathbb{Z}$ for X path connected. Today we observe that less trivial calculations get hard fast: the impressive edifice of homological algebra, which we now begin to develop properly, exists largely to facilitate such calculations. Here long exact sequences are a crucial tool.

Example Recall that for a ring R, and chain complex $C \in Ch_\bullet(R)$, we defined the homology

$$H_n(C) := \frac{\operatorname{Ker}(\partial_n : C_n \longrightarrow C_{n-1})}{\operatorname{Im}(\partial_{n+1} : C_{n+1} \longrightarrow C_n)}$$

as cycles (elements x with $\partial(x) = 0$) mod boundaries (y with x s.t. $y = \partial(x)$). This terminology originates in the case of the complex of abelian groups associated to a topological space X.

Let SC (singular chain complex) denote the functor

$$\underline{Top} \xrightarrow{S} \underline{SSet} = [\Delta^{op}, \underline{set}] \xrightarrow{V \circ -} [\Delta^{op}, \underline{Ab}] \xrightarrow{comp} Ch_\bullet(\mathbb{Z})$$

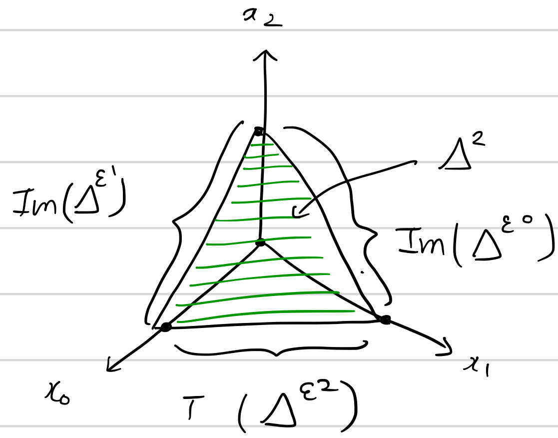
from Lecture 12, so that $SC(X) \in Ch_\bullet(\mathbb{Z})$ and $H_n(X) := H_n SC(X)$. Then cycles and boundaries in $SC(X)$ are "real" cycles and boundaries in X.

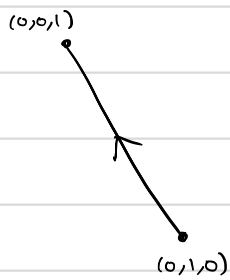
Example Let $\gamma : \Delta^1 \longrightarrow \mathbb{R}^n$ be a loop in $\mathbb{R}^n$, i.e. $\gamma(1,0) = \gamma(0,1)$. Clearly γ is a cycle, i.e. $\gamma \in Z_1 SC(\mathbb{R}^n)$. It is a boundary if we can find some linear combination of singular 2-simplices (= triangles) in $\mathbb{R}^n$ whose boundary is γ, i.e. if we can fill in the loop, in the way that the standard 2-simplex $\iota : \Delta^2 \hookrightarrow \mathbb{R}^3$ (ι the inclusion) fills in the cycle which is its boundary

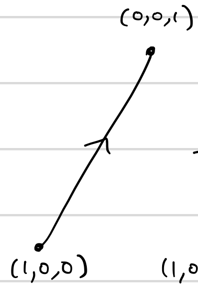
$\partial_2\Big($  $\Big) = \sum_{i=0}^{2} (-1)^i \, \iota \circ \Delta^{\varepsilon^i}$ where

$\Delta^2 \subseteq \mathbb{R}^3$

$$\left[\begin{array}{l} \varepsilon^0, \varepsilon^1, \varepsilon^2 : [1] \longrightarrow [2] \qquad \Delta^{\varepsilon^0} : \begin{array}{ccc} \Delta^1 & \longrightarrow & \Delta^2 \\ \cap\!| & & \cap\!| \\ \mathbb{R}^2 & & \mathbb{R}^3 \end{array} \\ \Delta^{\varepsilon^0}(x_0, x_1) = (0, x_0, x_1) \\ \Delta^{\varepsilon^1}(x_0, x_1) = (x_0, 0, x_1) \\ \Delta^{\varepsilon^2}(x_0, x_1) = (x_0, x_1, 0) \end{array}\right]$$



$=$ 

$-$ 

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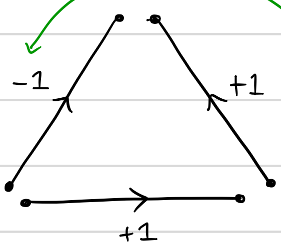
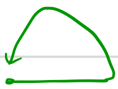
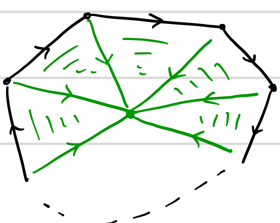
$=$ 

read the −1 as reversing the orientation, making this a loop 

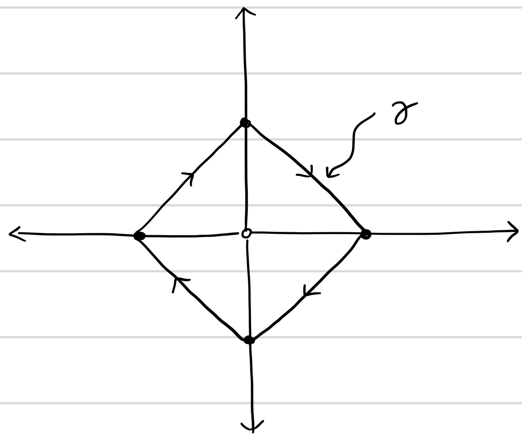
It is intuitively clear that any 1-cycle in $\mathbb{R}^n$ (= element of $Z_1 SC(\mathbb{R}^n)$) is a boundary, and thus that $H_1(\mathbb{R}^n) = 0$, by erecting cones over the pieces of the 1-cycle. A similar trick works in every dimension, so in fact



$$H_n(\mathbb{R}^m) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

But it's fairly obvious that proving some n-cycle $x \in Z_n SC(X)$ in a complicated space X is not a boundary (and thus gives a nonzero homology class $[x] \in H_n X$) is going to be hard, if we try to argue directly.

Example Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\gamma \in Z_1 SC(X)$ be as shown



(3.1)

While it is intuitively clear $[\gamma] \neq 0$ in $H_1 X$, how to prove it?
In fact one can show that for $n > 1$ and $F \subseteq \mathbb{R}^n$ a set of m points,

$$H_k(\mathbb{R}^n - F) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^m & k = n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

One of the most useful techniques for computing homology groups $H_n(X)$ are long exact sequences associated to subspaces $A \subseteq X$. One such long exact sequence is the main ingredient required to calculate $H_k(\mathbb{R}^n - F)$ (at least, in one standard approach, e.g. Dold's "Lectures on Algebraic Topology"). We turn now to a study of such long exact sequences.

<u>Def</u>

Def^{N} Let R be a ring, and $C \in Ch_\bullet(R)$. A <u>subcomplex</u> of C is a family of submodules $D_n \subseteq C_n$ s.t. for all n, $\partial_n(D_n) \subseteq D_{n-1}$, making $(D_n, \partial_n|_{D_n})$ a complex. In this case the quotients C_n/D_n also form a complex with differential $\overline{\partial}_n : C_n/D_n \longrightarrow C_{n-1}/D_{n-1}$, denoted C/D.

Ex 1 If $\iota : A \longrightarrow X$ is an injective continuous map, then $SC(\iota) : SCA \longrightarrow SCX$ is injective in each degree, and thus defines a subcomplex of SCX.

Ex 2 A subcomplex is precisely a subobject in $Ch_\bullet(R)$. Find the zero object in $Ch_\bullet(R)$ and prove the quotient maps define a morphism of complexes $C \longrightarrow C/D$ which is the cokernel of the inclusion $D \longrightarrow C$.

Def^{N} The <u>relative singular complex</u> of (X,A), with $A \subseteq X$ a subspace, is

$$SC(X,A) := SCX/SCA \in Ch_\bullet(\mathbb{Z}).$$

The <u>relative homology</u> is

$$H_n(X,A) := H_n SC(X,A). \qquad (4.1)$$

Def^{N} Let (C,∂) be a chain complex of R-modules. We say C is <u>exact</u> in degree $n \in \mathbb{Z}$ if $H_n C = 0$. We say C is <u>exact</u> if $H_n C = 0$ for all $n \in \mathbb{Z}$. A <u>short exact sequence</u> is a pair $\varphi : M \longrightarrow N$, $\psi : N \longrightarrow P$ of morphisms of R-modules with $\psi \circ \varphi = 0$ such that the complex

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0 \qquad (4.2)$$

is exact, that is, φ is injective, ψ is surjective and $\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)$.

<u>Def</u>

Def^N A <u>short exact sequence</u> in $Ch_\bullet(R)$ is a pair of morphisms of complexes $\varphi: C \longrightarrow D$, $\psi: D \longrightarrow E$ with $\psi \circ \varphi = 0$ (meaning $\psi_n \circ \varphi_n = 0$ for all n) and the morphisms $\varphi_n : C_n \longrightarrow D_n$, $\psi_n : D_n \longrightarrow E_n$ forming a short exact sequence for all $n \in \mathbb{Z}$,

$$0 \longrightarrow C_n \xrightarrow{\varphi_n} D_n \xrightarrow{\psi_n} E_n \longrightarrow 0.$$

In this case we write

$$0 \longrightarrow C \xrightarrow{\varphi} D \xrightarrow{\psi} E \longrightarrow 0 \qquad (5.1)$$

for the short exact sequence.

Ex 3 Prove in the setting of (4.2) that $P \cong N/M$, and in the setting of (5.1) that $E \cong D/C$. We'll return to this "properly" once we have understood abelian categories.

Theorem A Given a short exact sequence of complexes of R-modules

$$0 \longrightarrow C \xrightarrow{\varphi} D \xrightarrow{\psi} E \longrightarrow 0 \qquad (5.2)$$

there are R-linear maps $\{ \omega_n : H_n E \longrightarrow H_{n-1} C \}_{n \in \mathbb{Z}}$ called <u>connecting morphisms</u> with the property that the sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\omega_{n+1}"] & H_n C \arrow[r, "H_n\varphi"] & H_n D \arrow[r, "H_n\psi"] & H_n E \arrow[dll, "\omega_n"'] \\
& H_{n-1} C \arrow[r, "H_{n-1}\varphi"] & H_{n-1} D \arrow[r, "H_{n-1}\psi"] & H_{n-1} E \arrow[r, "\omega_{n-1}"] & \cdots
\end{tikzcd} \qquad (5.3)$$

is an exact complex.

This is called the <u>long exact homology sequence</u> associated to (5.1).
We will go into the proof next lecture, but first let us see how useful it is!

<u>Example</u> Let $A \subseteq X$ be a subspace with inclusion $\iota : A \to X$. By definition there is a short exact sequence

$$0 \longrightarrow SA \longrightarrow SX \longrightarrow SX/SA =: S(X,A) \longrightarrow 0 \qquad (6.1)$$

and hence a long exact sequence

$$\begin{aligned}
\cdots \longrightarrow\ & H_n A \longrightarrow H_n X \longrightarrow H_n(X,A) \\
\longrightarrow\ & H_{n-1} A \longrightarrow H_{n-1} X \longrightarrow H_{n-1}(X,A) \longrightarrow \cdots \\
\longrightarrow\ & H_1 A \longrightarrow H_1 X \longrightarrow H_1(X,A) \\
\longrightarrow\ & H_0 A \longrightarrow H_0 X \longrightarrow H_0(X,A) \longrightarrow 0.
\end{aligned} \qquad (6.2)$$

<u>Example</u> Let us apply (6.2) to $X = \mathbb{R}^n$, $A = \mathbb{R}^n - F$, F a finite number of points, $H_k(\mathbb{R}^n) = 0$ for $k > 0$ (which is easy) and $H_0(\mathbb{R}^n) \cong \mathbb{Z}$ (which we proved). Then (6.2) becomes

$$\begin{aligned}
\cdots \longrightarrow\ & H_k(\mathbb{R}^n - F) \longrightarrow 0 \longrightarrow H_k(\mathbb{R}^n, \mathbb{R}^n - F) \\
\longrightarrow\ & H_{k-1}(\mathbb{R}^n - F) \longrightarrow 0 \longrightarrow H_{k-1}(\mathbb{R}^n, \mathbb{R}^n - F) \longrightarrow \cdots \\
\cdots \longrightarrow\ & H_1(\mathbb{R}^n - F) \longrightarrow 0 \longrightarrow H_1(\mathbb{R}^n, \mathbb{R}^n - F) \\
\longrightarrow\ & H_0(\mathbb{R}^n - F) \longrightarrow \mathbb{Z} \longrightarrow H_0(\mathbb{R}^n, \mathbb{R}^n - F) \longrightarrow 0
\end{aligned} \qquad (6.3)$$

From exactness we deduce

$$H_k(\mathbb{R}^n, \mathbb{R}^n - F) \cong H_{k-1}(\mathbb{R}^n - F) \qquad k>1 \qquad (7.1)$$

So it suffices now to compute $H_k(\mathbb{R}^n, \mathbb{R}^n - F)$ which is done by reducing to the case $|F|=1$ where $H_k(\mathbb{R}^n, \mathbb{R}^n - F) \cong H_k(\mathbb{B}^n, \mathbb{S}^{n-1})$, and the latter is computed directly, as $H_k(\Delta^n, \partial\Delta^n)$. See p. 56 of Dold's "Lectures on Algebraic Topology". The upshot is $(n>1)$

$$H_k(\mathbb{R}^n, \mathbb{R}^n - F) = \begin{cases} 0 & k \neq n \\ \mathbb{Z}^{|F|} & k=n \end{cases}$$

Hence by (7.1) we have (as stated in (3.2) above)

$$H_k(\mathbb{R}^n - F) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{|F|} & k=n-1 \\ 0 & \text{otherwise} \end{cases}$$

In the case $n=2$, $F=\{(0,0)\}$ the generator of $H_1(\mathbb{R}^2 - \{0,0\})$ is the class of the cycle γ in (3.1). Since $H_0(\mathbb{R}^2 - \{0,0\}) \cong \mathbb{Z}$ (generated by the class of any point) and $H_k(\mathbb{R}^2 - \{0,0\}) = 0$ for $k>1$ this completely calculates the homology.

Conclusion The point of all this is not that I want you to be able to compute these homology groups yourself. I think it is important, however, that you have seen a sketch of a genuine application of the homological methods to the kind of problem they were invented to solve.