MAST90068 – Lecture 16

In this lecture we define the functor $Ext^q(-,-)$ which is functorial in two variables, and before doing so we have a brief digression on how to make sense of functors with more than one variable.

<u>Defⁿ</u> Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. The <u>product</u> $\mathcal{C}_1 \times \mathcal{C}_2$ is the category whose object class is $ob(\mathcal{C}_1) \times ob(\mathcal{C}_2)$, whose morphisms are

$$\mathrm{Hom}_{\mathcal{C}_1 \times \mathcal{C}_2}\big((a_1,a_2),(b_1,b_2)\big) := \mathrm{Hom}_{\mathcal{C}_1}(a_1,b_1) \times \mathrm{Hom}_{\mathcal{C}_2}(a_2,b_2)$$

and whose composition rule is, for $f_i : a_i \to b_i$, $g_i : b_i \to c_i$

$$(g_1,g_2) \circ (f_1,f_2) = (g_1 \circ f_1, g_2 \circ f_2).$$

<u>Lemma</u> $\mathcal{C}_1 \times \mathcal{C}_2$ is a category with $id_{(a,b)} = (id_a, id_b)$, and the obvious projections define functors $\pi_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ for $i \in \{1,2\}$.

<u>Ex 1</u> Prove the Lemma, and check that if $\mathcal{C}_1, \mathcal{C}_2$ are small then $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_1, \pi_2)$ is a product (in the limit sense) in <u>Cat</u>.

<u>Ex 2</u> What is the coproduct of $\mathcal{C}_1, \mathcal{C}_2$ in <u>Cat</u>?

<u>Defⁿ</u> A <u>bifunctor</u> F from a pair $(\mathcal{C}_1, \mathcal{C}_2)$ to a category $\mathcal{D}$ is the data of

(1) for each $A \in ob(\mathcal{C}_1)$ a functor $F(A,-) : \mathcal{C}_2 \to \mathcal{D}$

(2) for each $B \in ob(\mathcal{C}_2)$ a functor $F(-,B) : \mathcal{C}_1 \to \mathcal{D}$

satisfying

(3) for every pair $A \in ob(\mathcal{C}_1)$, $B \in ob(\mathcal{C}_2)$,

$$F(A,-)(B) = F(-,B)(A)$$

and we write $F(A,B)$ for this common object,

(4) for every pair of morphisms $f : A \to A'$ in $\mathcal{C}_1$ and $g : B \to B'$ in $\mathcal{C}_2$, the diagram

$$\begin{tikzcd}
F(A,B) \arrow[r, "{F(f,B)}"] \arrow[d, "{F(A,g)}"'] & F(A',B) \arrow[d, "{F(A',g)}"] \\
F(A,B') \arrow[r, "{F(f,B')}"'] & F(A',B')
\end{tikzcd}$$

commutes.

<u>Lemma</u> There is a bijection between functors $\mathcal{C}_1 \times \mathcal{C}_2 \xrightarrow{F} \mathcal{D}$ and bifunctors from $(\mathcal{C}_1, \mathcal{C}_2)$ to $\mathcal{D}$, which associates to F the data $\{F(A,-) : \mathcal{C}_2 \to \mathcal{D}\}_{A \in ob(\mathcal{C}_1)}$, $\{F(-,B) : \mathcal{C}_1 \to \mathcal{D}\}_{B \in ob(\mathcal{C}_2)}$.

<u>Ex 3</u> Prove the Lemma.

Henceforth we freely use whichever definition of bifunctor is convenient.

<u>Def^n</u> A <u>contravariant</u> functor $F : \mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{op} \to \mathcal{D}$.

<u>Example</u> Let $\mathcal{C}$ be a category. There is a bifunctor

$$H : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \underline{\mathrm{Set}}$$

defined by

$$H(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$$

and for $f : A \to A'$, $g : B \to B'$ in $\mathcal{C}$ by

$$H(A,g) : \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,B') \qquad x \longmapsto g \circ x$$
$$H(f,B) : \mathrm{Hom}_{\mathcal{C}}(A',B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,B) \qquad x \longmapsto x \circ f.$$

We often say $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is <u>contravariant</u> in the first variable (since the restrictions $\mathrm{Hom}_{\mathcal{C}}(-,B)$ are contravariant functors $\mathcal{C}^{op} \to \underline{\mathrm{Set}}$) and <u>covariant</u> in the second variable.

<u>Ex 4</u> For a ring R, check that the tensor product is a bifunctor

$$- \otimes_R - : \mathrm{Mod}R \times R\mathrm{Mod} \longrightarrow \underline{\mathrm{Ab}}$$

the diagram

$$\begin{tikzcd} K \arrow[r, "\sigma"] \arrow[dr, "\kappa"'] & L \arrow[d, "\lambda"] \\ & X \end{tikzcd}$$

commutative, $\lambda\sigma = \kappa$. Now let $\Delta = \varphi\alpha = \psi\beta$ be the diagonal of the square (6.2). Then the reader may easily prove

Proposition 6.1. *$(\Delta; \alpha, \beta)$ is the product of φ and ψ in $\mathfrak{C}/X$.* □

This means that α, β play the roles of p_1, p_2 in the definition of a product, when interpreted as morphisms $\alpha: \Delta \to \varphi$, $\beta: \Delta \to \psi$ in $\mathfrak{C}/X$. From this proposition we may readily deduce, from the propositions of Section 5, propositions about the pull-back and its evident generalization to a family, instead of a pair, of morphisms in $\mathfrak{C}$ with codomains X. We will prove one theorem about pull-backs in categories with zero objects which applies to the categories of interest in homological algebra. We recall first (Exercise 3.4) how we define the *kernel* of a morphism $\sigma: K \to L$ in a category with zero objects by means of a universal property. We say that $\mu: J \to K$ is a kernel of σ if (i) $\sigma\mu = 0$ and (ii) if $\alpha\tau = 0$ then τ factorizes as $\tau = \mu\tau_0$, with τ_0 unique. As usual, the kernel is essentially unique; we (sometimes) call J the *kernel object*. Notice that μ is monic, by virtue of the uniqueness of τ_0.

Theorem 6.2. *Let* (6.1) *be a pull-back diagram in a category $\mathfrak{C}$ with zero object. Then*

(i) *if (J, μ) is the kernel of β, $(J, \alpha\mu)$ is the kernel of φ;*

(ii) *if (J, ν) is the kernel of φ, ν may be factored as $\nu = \alpha\mu$ where (J, μ)* is the kernel of β.

Note that (ii) is superfluous if we know that every morphism in $\mathfrak{C}$ has a kernel. We show here, in particular, that β has a kernel if and only if φ has a kernel, and the kernel objects coincide.

Proof. (i)

$$\begin{tikzcd} J \arrow[r, equal] \arrow[d, "\mu"] & J \arrow[d, "\nu"] \\ Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

(6.1) is

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

Set $\nu = \alpha\mu$. We first show that ν is monomorphic; for μ and $\{\alpha, \beta\}$ are monomorphic, so $\{\alpha, \beta\}\,\mu = \{\nu, 0\}: J \to A \times B$ is monomorphic and hence, plainly, ν is monomorphic. Next we observe that $\varphi\nu = \varphi\alpha\mu = \psi\beta\mu = 0$.

Finally we take $\tau: Z \to A$ and show that if $\varphi\tau = 0$ then $\tau = \nu\tau_0$ for some τ_0. Since $\psi 0 = 0$, the pull-back property shows that there exists $\sigma: Z \to Y$ such that $\alpha\sigma = \tau$, $\beta\sigma = 0$. Since (J, μ) is the kernel of β, $\sigma = \mu\tau_0$, so that $\tau = \alpha\mu\tau_0 = \nu\tau_0$.

(ii) Since $\varphi\nu = 0$ we argue as in (i) that there exists $\mu: J \to Y$ with $\alpha\mu = \nu$, $\beta\mu = 0$. Since ν is a monomorphism, so is μ and we show that (J, μ) is the kernel of β. Let $\beta\tau = 0$, $\tau: Z \to Y$. Then $\varphi\alpha\tau = \psi\beta\tau = 0$, so $\alpha\tau = \nu\tau_0 = \alpha\mu\tau_0$. But $\beta\tau = \beta\mu\tau_0 = 0$, so that, $\{\alpha, \beta\}$ being a monomorphism, $\tau = \mu\tau_0$. □

In Chapter VIII we will refer back to this theorem as a very special case of a general result on commuting limits. We remark that the introduction of $A \times B$ in the proof was for convenience only. The argument is easily reformulated without invoking $A \times B$.

As examples of pull-backs, let us consider the categories $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{G}$. In $\mathfrak{S}$, let φ, ψ be embeddings of A, B as subsets of X; then $Y = A \cap B$ and α, β are also embeddings. In $\mathfrak{T}$ we could cite an example similar to that given for $\mathfrak{S}$; however there is also an interesting example when φ, say, is a fibre-map. Then β is also a fibre-map and is often called the fibre-map *induced by ψ from φ*. (Indeed, in general, the pull-back is sometimes called the *fibre-product*.) In $\mathfrak{G}$ we again have an example similar to that given for $\mathfrak{S}$; however there is a nice general description of Y as the subgroup of $A \times B$ consisting of those elements (a, b) such that $\varphi(a) = \psi(b)$.

The dual notion to that of a pull-back is that of a *push-out*. Thus, in (6.1), (φ, ψ) is the push-out of (α, β) in $\mathfrak{C}$ if and only if it is the pull-back of (α, β) in $\mathfrak{C}^{opp}$. The reader should have no difficulty in formulating an explicit universal property characterizing the push-out and dualizing the statements of this section. If α, β are embeddings (in $\mathfrak{S}$ or $\mathfrak{T}$) of $Y = A \cap B$ in A and B, then $X = A \cup B$. In $\mathfrak{G}$ we are led to the notion of *free product with amalgamations* [36].

We adopt for the push-out notational and terminological conventions analogous to those introduced for the pull-back.

Exercises:

6.1. Prove Proposition 6.1.

6.2. Given the commutative diagram in $\mathfrak{C}$

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[d, "\varphi_3"] \\ B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \end{tikzcd}$$

show that if both squares are pull-backs, so is the composite square. Show also that if the composite square is a pull-back and α_2 is monomorphic, then the left-hand square is a pull-back. Dualize these statements.

III. Extensions of Modules

In studying modules, as in studying any algebraic structures, the standard procedure is to look at submodules and associated quotient modules. The extension problem then appears quite naturally: given modules A, B (over a fixed ring Λ) what modules E may be constructed with submodule B and associated quotient module A? The set of equivalence classes of such modules E, written $E(A, B)$, may then be given an abelian group structure in a way first described by Baer [3]. It turns out that this group $E(A, B)$ is naturally isomorphic to a group $\mathrm{Ext}_\Lambda(A, B)$ obtained from A and B by the characteristic, indeed prototypical, methods of homological algebra. To be precise, $\mathrm{Ext}_\Lambda(A, B)$ is the value of the *first right derived* functor of $\mathrm{Hom}_\Lambda(-, B)$ on the module A, in the sense of Chapter IV.

In this chapter we study the homological and functorial properties of $\mathrm{Ext}_\Lambda(A, B)$. We show, in particular, that $\mathrm{Ext}_\Lambda(-, -)$ is balanced in the sense that $\mathrm{Ext}_\Lambda(A, B)$ is also the value of the first right derived functor of $\mathrm{Hom}_\Lambda(A, -)$ on the module B. Also, when $\Lambda = \mathbb{Z}$, so that A, B are abelian groups, we indicate how to compute the Ext groups; and prove a theorem of Stein-Serre showing how, for abelian groups of countable rank, the vanishing of $\mathrm{Ext}(A, \mathbb{Z})$ characterizes the free abelian groups A.

In view of the adjointness relation between the *tensor product* and Hom (see Theorem 7.2), it is natural to expect a similar theory for the tensor product and its first derived functors. This is given in the last two sections of the chapter.

1. Extensions

Let A, B be two Λ-modules. We want to consider all possible Λ-modules E such that B is a submodule of E and $E/B \cong A$. We then have a short exact sequence

$$B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$$

of Λ-modules; such a sequence is called an *extension* of A by B. We shall say that the extension $B \rightarrowtail E_1 \twoheadrightarrow A$ is *equivalent* to the extension $B \rightarrowtail E_2 \twoheadrightarrow A$ if there is a homomorphism $\xi: E_1 \to E_2$ such that the diagram

$$\begin{array}{ccccc} B & \rightarrowtail & E_1 & \twoheadrightarrow & A \\ \| & & \downarrow{\scriptstyle \xi} & & \| \\ B & \rightarrowtail & E_2 & \twoheadrightarrow & A \end{array}$$

is commutative. This relation plainly is transitive and reflexive. Since ξ is necessarily an isomorphism by Lemma I.1.1, it is symmetric, also.

The reader will notice that it would be possible to define an equivalence relation other than the one defined above: for example two extensions E_1, E_2 may be called equivalent if the modules E_1, E_2 are isomorphic, or they may be called equivalent if there exists a homomorphism $\xi: E_1 \to E_2$ inducing automorphisms in both A and B. In our definition of equivalence we insist that the homomorphism $\xi: E_1 \to E_2$ induces the *identity* in both A and B. We refer the reader to Exercise 1.1 which shows that the different definitions of equivalence are indeed different notions. The reason we choose our definition will become clear with Theorem 1.4 and Corollary 2.5.

We denote the set of equivalence classes of extensions of A by B by $E(A, B)$. Obviously $E(A, B)$ contains at least one element: The Λ-module $A \oplus B$, together with the maps ι_B, π_A, yields an extension

$$B \overset{\iota_B}{\rightarrowtail} A \oplus B \overset{\pi_A}{\twoheadrightarrow} A\,. \tag{1.1}$$

The map $\iota_A: A \to A \oplus B$ satisfies the equation $\pi_A \iota_A = 1_A$ and the map $\pi_B: A \oplus B \to B$ the equation $\pi_B \iota_B = 1_B$. Because of the existence of such maps we call any extension equivalent to (1.1) a *split extension* of A by B.

Our aim is now to make $E(-,-)$ into a functor; we therefore have to define induced maps. The main part of the work is achieved by the following lemmas.

Lemma 1.1. *The square*

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\psi} & X \end{array} \tag{1.2}$$

is a pull-back diagram if and only if the sequence

$$0 \to Y \xrightarrow{\{\alpha, \beta\}} A \oplus B \xrightarrow{\langle \varphi, -\psi \rangle} X$$

is exact.

Proof. We have to show that the universal property of the pull-back of (φ, ψ) is the same as the universal property of the kernel of $\langle \varphi, -\psi \rangle$. But it is plain that two maps $\gamma: Z \to A$ and $\delta: Z \to B$ make the square

$$\begin{array}{ccc} Z & \xrightarrow{\gamma} & A \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\psi} & X \end{array}$$

commutative if and only if they induce a map $\{\gamma,\delta\}: Z \to A \oplus B$ such that $\langle\varphi, -\psi\rangle \circ \{\gamma,\delta\} = 0$. The universal property of the kernel asserts the existence of a unique map $\zeta: Z \to Y$ with $\{\alpha,\beta\} \circ \zeta = \{\gamma,\delta\}$. The universal property of the pull-back asserts the existence of a unique map $\zeta: Z \to Y$ with $\alpha \circ \zeta = \gamma$ and $\beta \circ \zeta = \delta$. □

Lemma 1.2. *If the square* (1.2) *is a pull-back diagram, then*

just proven (i) *β induces* $\ker\alpha \xrightarrow{\sim} \ker\psi$;

(ii) *if ψ is an epimorphism, then so is α.*

Proof. Part (i) has been proved in complete generality in Theorem II.6.4. For part (ii) we consider the sequence $0 \to Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{\langle\varphi,-\psi\rangle} X$, which is exact by Lemma 1.1. Suppose $a \in A$. Since ψ is epimorphic there exists $b \in B$ with $\varphi a = \psi b$, whence it follows that $(a, b) \in \ker\langle\varphi, -\psi\rangle = \operatorname{im}\{\alpha,\beta\}$ by exactness. Thus there exists $y \in Y$ with $a = \alpha y$ (and $b = \beta y$). Hence α is epimorphic. □

We now prove a partial converse of Lemma 1.2 (i).

Lemma 1.3. *Let*

$$\begin{tikzcd} B \arrow[r, tail, "\kappa'"] \arrow[d, equal] & E' \arrow[r, two heads, "\nu'"] \arrow[d, "\xi"] & A' \arrow[d, "\alpha"] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

be a commutative diagram with exact rows. Then the right-hand square is a pull-back diagram.

Proof. Let

$$\begin{tikzcd} P \arrow[r, "\varepsilon"] \arrow[d, "\varphi"] & A' \arrow[d, "\alpha"] \\ E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

be a pull-back diagram. By Lemma 1.2 ε is epimorphic and φ induces an isomorphism $\ker\varepsilon \cong B$. Hence we obtain an extension

$$B \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A'.$$

By the universal property of P there exists a map $\zeta: E' \to P$, such that $\varphi\zeta = \xi$, $\varepsilon\zeta = \nu'$. Since ζ induces the identity in both A' and B, ζ is an isomorphism by Lemma I.1.1. □

We leave it to the reader to prove the duals of Lemmas 1.1, 1.2, 1.3. In the sequel we shall feel free to refer to these lemmas when we require either their statements or the dual statements.

Let $\alpha: A' \to A$ be a homomorphism and let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ be a representative of an element in $E(A, B)$. Consider the diagram

$$\begin{tikzcd} & E^\alpha \arrow[r, dotted, "\nu'"] \arrow[d, dotted, "\xi"] & A' \arrow[d, "\alpha"] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

where $(E^\alpha; \nu', \xi)$ is the pull-back of (α, ν). By Lemma 1.2 we obtain an extension $B \rightarrowtail E^\alpha \overset{\nu'}{\twoheadrightarrow} A'$. Thus we can define our induced map

$$\alpha^* : E(A, B) \to E(A', B)$$

by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ the class of $B \rightarrowtail E^\alpha \twoheadrightarrow A'$. Plainly this definition is independent of the chosen representative $B \rightarrowtail E \twoheadrightarrow A$.

We claim that this definition of $E(\alpha, B) = \alpha^*$ makes $E(-, B)$ into a contravariant functor. Indeed it is plain that for $\alpha = 1_A : A \to A$ the induced map is the identity in $E(A, B)$. Let $\alpha' : A'' \to A'$ and $\alpha : A' \to A$. In order to show that $E(\alpha \circ \alpha', B) = E(\alpha', B) \circ E(\alpha, B)$, we have to prove that in the diagram

$$\begin{tikzcd} (E^\alpha)^{\alpha'} \arrow[r, dotted] \arrow[d, dotted] & A'' \arrow[d, "\alpha'"] \\ E^\alpha \arrow[r, dotted] \arrow[d, dotted] & A' \arrow[d, "\alpha"] \\ E \arrow[r, "\nu"] & A \end{tikzcd}$$

where each square is a pull-back, the composite square is the pull-back of $(\nu, \alpha \circ \alpha')$. But this follows readily from the universal property of the pull-back.

Now let $\beta: B \to B'$ be a homomorphism, and let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ again be a representative of an element in $E(A. B)$. We consider the diagram

$$\begin{tikzcd} B \arrow[r, tail, "\kappa"] \arrow[d, "\beta"] & E \arrow[r, two heads, "\nu"] \arrow[d, dotted, "\xi"] & A \\ B' \arrow[r, dotted, "\kappa'"] & E_\beta & \end{tikzcd}$$

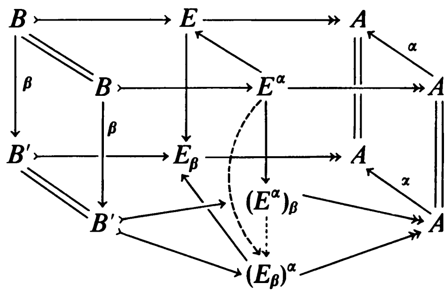
where $(E_\beta; \kappa', \xi)$ is the push-out of (β, κ). The dual of Lemma 1.2 shows that we obtain an extension $B' \rightarrowtail E_\beta \twoheadrightarrow A$. We then can define

$$\beta_* : E(A, B) \to E(A, B')$$

by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ the class of $B' \rightarrowtail E_\beta \twoheadrightarrow A$. As above one easily proves that this definition of $E(B, \beta) = \beta_*$ makes $E(A, -)$ into a covariant functor. Indeed, we even assert:

Theorem 1.4. *$E(-, -)$ is a bifunctor from the category of Λ-modules to the category of sets. It is contravariant in the first and covariant in the second variable.*

Proof. It remains to check that $\beta_* \alpha^* = \alpha_* \beta^* : E(A, B) \to E(A', B')$. We can construct the following (3-dimensional) commutative diagram, using pull-backs and push-outs.



We have to show the existence of $(E^\alpha)_\beta \to (E_\beta)^\alpha$ such that the diagram remains commutative. We first construct $E^\alpha \to (E_\beta)^\alpha$ satisfying the necessary commutativity relations. Since $E^\alpha \to E \to E_\beta \to A$ coincides with $E^\alpha \to A' \to A$, we do indeed find $E^\alpha \to (E_\beta)^\alpha$ such that $E^\alpha \to (E_\beta)^\alpha \to E_\beta$ coincides with $E^\alpha \to E \to E_\beta$ and $E^\alpha \to (E_\beta)^\alpha \to A'$ coincides with $E^\alpha \to A'$. It remains to check that $B \to E^\alpha \to (E_\beta)^\alpha$ coincides with $B \to B' \to (E_\beta)^\alpha$. By the uniqueness of the map into the pull-back it suffices to check that $B \to E^\alpha \to (E_\beta)^\alpha \to E_\beta$ coincides with $B \to B' \to (E_\beta)^\alpha \to E_\beta$ and $B \to E^\alpha \to (E_\beta)^\alpha \to A'$ coincides with $B \to B' \to (E_\beta)^\alpha \to A'$, and these facts follow from the known commutativity relations. Since $B \to E^\alpha \to (E_\beta)^\alpha$ coincides with $B \to B' \to (E_\beta)^\alpha$ we find $(E^\alpha)_\beta \to (E_\beta)^\alpha$ such that $B' \to (E^\alpha)_\beta \to (E_\beta)^\alpha$ coincides with $B' \to (E_\beta)^\alpha$ and $E^\alpha \to (E^\alpha)_\beta \to (E_\beta)^\alpha$ coincides with $E^\alpha \to (E_\beta)^\alpha$. It only remains to show that $(E^\alpha)_\beta \to (E_\beta)^\alpha \to A'$ coincides with $(E^\alpha)_\beta \to A'$. Again, uniqueness considerations allow us merely to prove that $B' \to (E^\alpha)_\beta \to (E_\beta)^\alpha \to A'$ coincides with $B' \to (E^\alpha)_\beta \to A'$, and $E^\alpha \to (E^\alpha)_\beta \to (E_\beta)^\alpha \to A'$ coincides with $E^\alpha \to (E^\alpha)_\beta \to A'$. Since these facts, too, follow from the known commutativity relations, the theorem is proved. $\square$

Exercises:

1.1. Show that the following two extensions are non-equivalent

$$\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}_3, \qquad \mathbb{Z} \overset{\mu'}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon'}{\twoheadrightarrow} \mathbb{Z}_3$$

where $\mu = \mu'$ is multiplication by 3, $\varepsilon(1) = 1 \pmod 3$ and $\varepsilon'(1) = 2 \pmod 3$.

1.2. Compute $E(\mathbb{Z}_p, \mathbb{Z})$, p prime.

1.3. Prove the duals of Lemmas 1.1, 1.2, 1.3.
1.4. Show that the class of the split extension in $E(A, B)$ is preserved under the induced maps.
1.5. Prove: If P is projective, $E(P, B)$ contains only one element.
1.6. Prove: If I is injective, $E(A, I)$ contains only one element.
1.7. Show that $E(A, B_1 \oplus B_2) \cong E(A, B_1) \times E(A, B_2)$. Is there a corresponding formula with respect to the first variable?
1.8. Prove Theorem 1.4 using explicit constructions of pull-back and push-out.

2. The Functor Ext

In the previous section we have defined a bifunctor $E(-,-)$ from the category of Λ-modules to the categories of sets. In this section we shall define another bifunctor $\mathrm{Ext}_\Lambda(-,-)$ to the category of abelian groups, and eventually compare the two.

A short exact sequence $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of Λ-modules with P projective is called a *projective presentation* of A. By Theorem I.2.2 such a presentation induces for a Λ-module B an exact sequence

$$\mathrm{Hom}_\Lambda(A, B) \overset{\varepsilon^*}{\rightarrowtail} \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(R, B). \tag{2.1}$$

To the modules A and B, and to the chosen projective presentation of A we therefore can associate the abelian group

$$\mathrm{Ext}_\Lambda^\varepsilon(A, B) = \operatorname{coker}(\mu^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(R, B)).$$

The superscript ε is to remind the reader that the group is defined via a particular projective presentation of A. An element in $\mathrm{Ext}_\Lambda^\varepsilon(A, B)$ may be represented by a homomorphism $\varphi : R \to B$. The element represented by $\varphi : R \to B$ will be denoted by $[\varphi] \in \mathrm{Ext}_\Lambda^\varepsilon(A, B)$. Then $[\varphi_1] = [\varphi_2]$ if and only if $\varphi_1 - \varphi_2$ extends to P.

Clearly a homomorphism $\beta : B \to B'$ will map the sequence (2.1) into the corresponding sequence for B'. We thus get an induced map $\beta_* : \mathrm{Ext}_\Lambda^\varepsilon(A, B) \to \mathrm{Ext}_\Lambda^\varepsilon(A, B')$, which is easily seen to make $\mathrm{Ext}_\Lambda^\varepsilon(A, -)$ into a functor.

Next we will show that for two different projective presentations of A we obtain the "same" functor. Let $R' \overset{\mu'}{\rightarrowtail} P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ be projective presentations of A', A respectively. Let $\alpha : A' \to A$ be a homomorphism. Since P' is projective, there is a homomorphism $\pi : P' \to P$, inducing $\sigma : R' \to R$ such that the following diagram is commutative:

$$\begin{tikzcd} R' \arrow[r, tail, "\mu'"] \arrow[d, "\sigma"] & P' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\pi"] & A' \arrow[d, "\alpha"] \\ R \arrow[r, tail, "\mu"] & P \arrow[r, two heads, "\varepsilon"] & A \end{tikzcd}$$

We sometimes say that π *lifts* α.