Theorem 2.1. *Let $B' \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\to} B''$ be an exact sequence of Λ-modules. For every Λ-module A the induced sequence*

$$0 \to \mathrm{Hom}_\Lambda(A, B') \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(A, B'')$$

is exact.

Proof. First we show that μ_* is injective.

Assume that $\mu\varphi$ in the diagram

$$\begin{tikzcd} A \arrow[d, "\varphi"] & & \\ B' \arrow[r, tail, "\mu"] & B \arrow[r, "\varepsilon"] & B'' \end{tikzcd}$$

is the zero map. Since $\mu: B' \rightarrowtail B$ is injective this implies that $\varphi: A \to B'$ is the zero map, so μ_* is injective.

Next we show that $\ker\varepsilon_* \supset \operatorname{im}\mu_*$. Consider the above diagram. A map in $\operatorname{im}\mu_*$ is of the form $\mu\varphi$. Plainly $\varepsilon\mu\varphi$ is the zero map, since $\varepsilon\mu$ already is. Finally we show that $\operatorname{im}\mu_* \supset \ker\varepsilon_*$. Consider the diagram

$$\begin{tikzcd} & A \arrow[d, "\psi"] & \\ B' \arrow[r, tail, "\mu"] & B \arrow[r, "\varepsilon"] & B''. \end{tikzcd}$$

We have to show that if $\varepsilon\psi$ is the zero map, then ψ is of the form $\mu\varphi$ for some $\varphi: A \to B'$. But, if $\varepsilon\psi = 0$ the image of ψ is contained in $\ker\varepsilon = \operatorname{im}\mu$. Since μ is injective, ψ gives rise to a (unique) map $\varphi: A \to B'$ such that $\mu\varphi = \psi$. □

We remark that even in case ε is surjective the induced map ε_* is *not* surjective in general (see Exercise 2.1).

Theorem 2.2. *Let $A' \overset{\mu}{\to} A \overset{\varepsilon}{\twoheadrightarrow} A''$ be an exact sequence of Λ-modules. For every Λ-module B the induced sequence*

$$0 \to \mathrm{Hom}_\Lambda(A'', B) \xrightarrow{\varepsilon^*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(A', B)$$

is exact.

The proof is left to the reader. □

Notice that even in case μ is injective μ^* is *not* surjective in general (see Exercise 2.2).

We finally remark that Theorem 2.1 provides a universal characterization of $\ker\varepsilon$ (in the sense of Sections II.5 and II.6): To every homomorphism $\varphi: A \to B$ with $\varepsilon_*(\varphi) = \varepsilon\varphi: A \to B''$ the zero map there exists a unique homomorphism $\varphi': A \to B'$ with $\mu_*(\varphi') = \mu\varphi' = \varphi$. Similarly Theorem 2.2 provides a universal characterization of $\operatorname{coker}\mu$.

1.3. Prove the duals of Lemmas 1.1, 1.2, 1.3.
1.4. Show that the class of the split extension in $E(A,B)$ is preserved under the induced maps.
1.5. Prove: If P is projective, $E(P,B)$ contains only one element.
1.6. Prove: If I is injective, $E(A,I)$ contains only one element.
1.7. Show that $E(A,B_1 \oplus B_2) \cong E(A,B_1) \times E(A,B_2)$. Is there a corresponding formula with respect to the first variable?
1.8. Prove Theorem 1.4 using explicit constructions of pull-back and push-out.

2. The Functor Ext

In the previous section we have defined a bifunctor $E(-,-)$ from the category of Λ-modules to the categories of sets. In this section we shall define another bifunctor $\mathrm{Ext}_\Lambda(-,-)$ to the category of abelian groups, and eventually compare the two.

A short exact sequence $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of Λ-modules with P projective is called a *projective presentation* of A. By Theorem I.2.2 such a presentation induces for a Λ-module B an exact sequence

$$\mathrm{Hom}_\Lambda(A,B) \overset{\varepsilon^*}{\rightarrowtail} \mathrm{Hom}_\Lambda(P,B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(R,B). \tag{2.1}$$

To the modules A and B, and to the chosen projective presentation of A we therefore can associate the abelian group

$$\mathrm{Ext}_\Lambda^\varepsilon(A,B) = \mathrm{coker}(\mu^* : \mathrm{Hom}_\Lambda(P,B) \to \mathrm{Hom}_\Lambda(R,B)).$$

The superscript ε is to remind the reader that the group is defined via a particular projective presentation of A. An element in $\mathrm{Ext}_\Lambda^\varepsilon(A,B)$ may be represented by a homomorphism $\varphi : R \to B$. The element represented by $\varphi : R \to B$ will be denoted by $[\varphi] \in \mathrm{Ext}_\Lambda^\varepsilon(A,B)$. Then $[\varphi_1] = [\varphi_2]$ if and only if $\varphi_1 - \varphi_2$ extends to P.

Clearly a homomorphism $\beta : B \to B'$ will map the sequence (2.1) into the corresponding sequence for B'. We thus get an induced map $\beta_* : \mathrm{Ext}_\Lambda^\varepsilon(A,B) \to \mathrm{Ext}_\Lambda^\varepsilon(A,B')$, which is easily seen to make $\mathrm{Ext}_\Lambda^\varepsilon(A,-)$ into a functor.

Next we will show that for two different projective presentations of A we obtain the "same" functor. Let $R' \overset{\mu'}{\rightarrowtail} P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ be projective presentations of A', A respectively. Let $\alpha : A' \to A$ be a homomorphism. Since P' is projective, there is a homomorphism $\pi : P' \to P$, inducing $\sigma : R' \to R$ such that the following diagram is commutative:

$$\begin{tikzcd} R' \arrow[r, tail, "\mu'"] \arrow[d, "\sigma"] & P' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\pi"] & A' \arrow[d, "\alpha"] \\ R \arrow[r, tail, "\mu"] & P \arrow[r, two heads, "\varepsilon"] & A \end{tikzcd}$$

We sometimes say that π *lifts* α.

Clearly π, together with σ, will induce a map

$$\pi^* : \mathrm{Ext}_\Lambda^\varepsilon(A, B) \to \mathrm{Ext}_\Lambda^{\varepsilon'}(A', B)$$

which plainly is natural in B. Thus every π gives rise to a natural transformation from $\mathrm{Ext}_\Lambda^\varepsilon(A, -)$ into $\mathrm{Ext}_\Lambda^{\varepsilon'}(A', -)$. In the following lemma we prove that this natural transformation depends only on $\alpha : A' \to A$ and not on the chosen $\pi : P' \to P$ lifting α.

Lemma 2.1. *π^* does not depend on the chosen $\pi : P' \to P$ but only on $\alpha : A' \to A$.*

Proof. Let $\pi_i : P' \to P$, $i = 1, 2$, be two homomorphisms lifting α and inducing $\sigma_i : R' \to R$, so that the following diagram is commutative for $i = 1, 2$

$$\begin{tikzcd}
R' \arrow[r, tail, "\mu'"] \arrow[d, "\sigma_i"] & P' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\pi_i"] & A' \arrow[d, "\alpha"] \\
R \arrow[r, tail, "\mu"] & P \arrow[r, two heads, "\varepsilon"] & A
\end{tikzcd}$$

Consider $\pi_1 - \pi_2$; since π_1, π_2 induce the same map $\alpha : A' \to A$, $\pi_1 - \pi_2$ factors through a map $\tau : P' \to R$, such that $\pi_1 - \pi_2 = \mu\tau$. It follows that $\sigma_1 - \sigma_2 = \tau\mu'$. Thus, if $\varphi : R \to B$ is a representative of the element $[\varphi] \in \mathrm{Ext}_\Lambda^\varepsilon(A, B)$, we have $\pi_1^*[\varphi] = [\varphi\sigma_1] = [\varphi\sigma_2 + \varphi\tau\mu'] = [\varphi\sigma_2] = \pi_2^*[\varphi]$. □

To stress the independence from the choice of π we shall call the natural transformation $(\alpha; P', P) : \mathrm{Ext}_\Lambda^\varepsilon(A, -) \to \mathrm{Ext}_\Lambda^{\varepsilon'}(A', -)$, instead of π^*. Let $\alpha' : A'' \to A'$ and $\alpha : A' \to A$ be two homomorphisms and $R'' \rightarrowtail P'' \twoheadrightarrow A''$, $R' \rightarrowtail P' \twoheadrightarrow A'$, $R \rightarrowtail P \to A$ projective presentations of A'', A', A respectively. Let $\pi' : P'' \to P'$ lift α' and $\pi : P' \to P$ lift α. Then $\pi \circ \pi' : P'' \to P$ lifts $\alpha \circ \alpha'$; whence it follows that

$$(\alpha'; P'', P') \circ (\alpha; P', P) = (\alpha \circ \alpha'; P'', P). \tag{2.2}$$

Also, we have

$$(1_A; P, P) = 1. \tag{2.3}$$

This yields a proof of

Corollary 2.2. *Let $R \rightarrowtail P \overset{\varepsilon}{\twoheadrightarrow} A$ and $R' \rightarrowtail P' \overset{\varepsilon'}{\twoheadrightarrow} A$ be two projective presentations of A. Then*

$$(1_A; P', P) : \mathrm{Ext}_\Lambda^\varepsilon(A, -) \to \mathrm{Ext}_\Lambda^{\varepsilon'}(A, -)$$

is a natural equivalence.

Proof. Let $\pi : P \to P'$ and $\pi' : P' \to P$ both lift $1_A : A \to A$. By formulas (2.2) and (2.3) we obtain $(1_A; P, P') \circ (1_A; P', P) = (1_A; P, P) = 1 : \mathrm{Ext}_\Lambda^\varepsilon(A, -) \to \mathrm{Ext}_\Lambda^\varepsilon(A, -)$. Analogously, $(1_A; P', P) \circ (1_A; P, P') = 1$, whence the assertion. □

By this natural equivalence we are allowed to drop the superscript ε and to write, simply, $\mathrm{Ext}_\Lambda(A, B)$.

Of course, we want to make $\mathrm{Ext}_\Lambda(-, B)$ into a functor. It is obvious by now that given $\alpha : A' \to A$ we can define an induced map α^* as follows: Choose projective presentations $R' \rightarrowtail P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \rightarrowtail P \overset{\varepsilon}{\twoheadrightarrow} A$ of A', A respectively, and let $\alpha^* = (\alpha; P', P) : \mathrm{Ext}^\varepsilon_\Lambda(A, B) \to \mathrm{Ext}^{\varepsilon'}_\Lambda(A', B)$. Formulas (2.2), (2.3) establish the facts that this definition is compatible with the natural equivalences of Corollary 2.2 and that $\mathrm{Ext}_\Lambda(-, B)$ becomes a (contravariant) functor. We leave it to the reader to prove the *bifunctoriality* part in the following theorem.

Theorem 2.3. $\mathrm{Ext}_\Lambda(-, -)$ *is a bifunctor from the category of Λ-modules to the category of abelian groups. It is contravariant in the first, and covariant in the second variable.* □

Instead of regarding $\mathrm{Ext}_\Lambda(A, B)$ as an abelian group, we clearly can regard it just as a set. We thus obtain a *set*-valued bifunctor which – for convenience – we shall still call $\mathrm{Ext}_\Lambda(-, -)$.

Theorem 2.4. *There is a natural equivalence of set-valued bifunctors* $\eta : E(A, B) \xrightarrow{\sim} \mathrm{Ext}_\Lambda(A, B)$.

Proof. We first define an isomorphism of sets

$$\eta : E(A, B) \xrightarrow{\sim} \mathrm{Ext}^\varepsilon_\Lambda(A, B),$$

natural in B, where $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ is a fixed projective presentation of A. We will then show that η is natural in A.

Given an element in $E(A, B)$, represented by the extension $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$, we form the diagram

$$\begin{tikzcd} R \arrow[r, tail, "\mu"] \arrow[d, dotted, "\psi"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, dotted, "\varphi"] & A \arrow[d, equal] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

The homomorphism $\psi : R \to B$ defines an element $[\psi] \in \mathrm{Ext}^\varepsilon_\Lambda(A, B) = \mathrm{coker}(\mu^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(R, B))$. We claim that this element does not depend on the particular $\varphi : P \to E$ chosen. Thus let $\varphi_i : P \to E$, $i = 1, 2$, be two maps inducing $\psi_i : R \to B$, $i = 1, 2$. Then $\varphi_1 - \varphi_2$ factors through $\tau : P \to B$, i.e., $\varphi_1 - \varphi_2 = \kappa\tau$. It follows that $\psi_1 - \psi_2 = \tau\mu$, whence $[\psi_1] = [\psi_2 + \tau\mu] = [\psi_2]$.

Since two representatives of the same element in $E(A, B)$ obviously induce the same element in $\mathrm{Ext}^\varepsilon_\Lambda(A, B)$, we have defined a map $\eta : E(A, B) \to \mathrm{Ext}^\varepsilon_\Lambda(A, B)$. We leave it to the reader to prove the naturality of η with respect to B.

Conversely, given an element in $\mathrm{Ext}^\varepsilon_\Lambda(A, B)$, we represent this element by a homomorphism $\psi : R \to B$. Taking the push-out of (ψ, μ) we obtain the diagram

$$\begin{tikzcd}
R \arrow[r, tail, "\mu"] \arrow[d, "\psi"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, dotted, "\varphi"] & A \arrow[d, equal] \\
B \arrow[r, dotted, "\kappa"] & E \arrow[r, dotted, "\nu"] & A
\end{tikzcd}$$

By the dual of Lemma 1.2 the bottom row $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ is an extension. We claim that the equivalence class of this extension is independent of the particular representative $\psi : R \to B$ chosen. Indeed another representative $\psi' : R \to B$ has the form $\psi' = \psi + \tau\mu$ where $\tau : P \to B$. The reader may check that the diagram

$$\begin{tikzcd}
R \arrow[r, tail, "\mu"] \arrow[d, "\psi'"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\varphi'"] & A \arrow[d, equal] \\
B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A
\end{tikzcd}$$

with $\varphi' = \varphi + \kappa\tau$ is commutative. By the dual of Lemma 1.3 the left hand square is a push-out diagram, whence it follows that the extension we arrive at does not depend on the representative. We thus have defined a map

$$\xi : \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B) \to E(A, B)$$

which is easily seen to be natural in B.

Using Lemma 1.3 it is easily proved that η, ξ are inverse to each other. We thus have an equivalence

$$\eta : E(A, B) \xrightarrow{\sim} \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B)$$

which is natural in B.

Note that η might conceivably depend upon the projective presentation of A. However we show that this cannot be the case by the following (3-dimensional) diagram, which shows also the naturality of η in A.

$$\begin{tikzcd}
R \arrow[rr, tail] \arrow[dd] & & P \arrow[rr, two heads, "\varepsilon"] \arrow[dd] & & A \arrow[dd, equal] & \\
& R' \arrow[ul] \arrow[rr, tail, crossing over] \arrow[dd, dotted, "\psi"] & & P' \arrow[ul] \arrow[rr, two heads, crossing over] \arrow[dd, dotted, "\varphi"] & & A' \arrow[ul, "\alpha"'] \arrow[dd, equal] \\
B \arrow[rr, tail] \arrow[dr, equal] & & E \arrow[rr, two heads] & & A & \\
& B \arrow[rr, tail] & & E^{\alpha} \arrow[ul] \arrow[rr, two heads] & & A' \arrow[ul, "\alpha"']
\end{tikzcd}$$

E^{α} is the pull-back of $E \to A$ and $A' \to A$. We have to show the existence of homomorphisms $\varphi : P' \to E^{\alpha}$, $\psi : R' \to B$ such that all faces are commutative. Since the maps $P' \to E \to A$ and $P' \to A' \to A$ agree they define a homomorphism $\varphi : P' \to E^{\alpha}$, into the pull-back. Then φ induces

$\psi : R' \to B$, and trivially all faces are commutative. (To see that $R' \to R \to B$ coincides with ψ, compose each with $B \rightarrowtail E$.) We therefore arrive at a commutative diagram

$$\begin{tikzcd} E(A,B) \arrow[r, "\alpha^*"] \arrow[d, shift right, "\eta"'] & E(A',B) \arrow[d, shift right, "\eta"'] \\ \mathrm{Ext}_A^{\varepsilon}(A,B) \arrow[r, "\alpha^*"] \arrow[u, shift right, "\xi"'] & \mathrm{Ext}_A^{\varepsilon'}(A',B) \arrow[u, shift right, "\xi"'] \end{tikzcd}$$

For $A' = A$, $\alpha = 1_A$ this shows that η is independent of the chosen projective presentation. In general it shows that η and ξ are natural in A. $\square$

Corollary 2.5. *The set $E(A, B)$ of equivalence classes of extensions has a natural abelian group structure.*

Proof. This is obvious, since $\mathrm{Ext}_A(A, B)$ carries a natural abelian group structure and since $\eta : E(-,-) \xrightarrow{\sim} \mathrm{Ext}_A(-,-)$ is a natural equivalence. $\square$

We leave as exercises (see Exercises 2.5 to 2.7) the direct description of the group structure in $E(A, B)$. However we shall exhibit here the neutral element of this group. Consider the diagram

$$\begin{tikzcd} R \arrow[r, "\mu"] \arrow[d, "\psi"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\varphi"] & A \arrow[d, equal] \\ B \arrow[r, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

The extension $B \rightarrowtail E \twoheadrightarrow A$ represents the neutral element in $E(A, B)$ if and only if $\psi : R \to B$ is the restriction of a homomorphism $\tau : P \to B$, i.e., if $\psi = \tau\mu$. The map $(\varphi - \kappa\tau)\mu : R \to E$ therefore is the zero map. so that $\varphi - \kappa\tau$ factors through A, defining a map $\sigma : A \to E$ with $\varphi - \kappa\tau = \sigma\varepsilon$. Since $\nu(\varphi - \kappa\tau) = \varepsilon$, σ is a right inverse to ν. Thus the extension $B \rightarrowtail E \twoheadrightarrow A$ splits. Conversely if $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ splits, the left inverse of κ is a map $E \to B$ which if composed with $\varphi : P \to E$ yields τ.

We finally note

Proposition 2.6. *If P is projective and I injective, then* $\mathrm{Ext}_A(P, B) = 0 = \mathrm{Ext}_A(A, I)$ *for all A-modules A, B.*

Proof. By Theorem 2.4 $\mathrm{Ext}_A(P, B)$ is in one-to-one correspondence with the set $E(P, B)$, consisting of classes of extensions of the form $B \rightarrowtail E \twoheadrightarrow P$. By Lemma I.4.5 short exact sequences of this form split. Hence $E(P, B)$ contains only one element, the zero element. For the other assertion one proceeds dually. $\square$

Of course, we could prove this proposition directly, without involving Theorem 2.4.

Exercises:

2.1. Prove that $\operatorname{Ext}_\Lambda(-,-)$ is a bifunctor.

2.2. Suppose A is a right Γ-left Λ-bimodule. Show that $\operatorname{Ext}_\Lambda(A,B)$ has a left-Γ-module structure which is natural in B.

2.3. Suppose B is a right Γ-left Λ-bimodule. Show that $\operatorname{Ext}_\Lambda(A,B)$ has a right Γ-module structure, which is natural in A.

2.4. Suppose Λ commutative. Show that $\operatorname{Ext}_\Lambda(A,B)$ has a natural (in A and B) Λ-module structure.

2.5. Show that one can define an addition in $E(A,B)$ as follows: Let $B \rightarrowtail E_1 \twoheadrightarrow A$, $B \rightarrowtail E_2 \twoheadrightarrow A$ be representatives of two elements ξ_1, ξ_2 in $E(A,B)$. Let $\Delta_B : B \to B \oplus B$ be the map defined by $\Delta_B(b) = (b,b)$, $b \in B$, and let $\nabla_A : A \oplus A \to A$ be the map defined by $\nabla_A(a_1,a_2) = a_1 + a_2$, $a_1, a_2 \in A$. Define the sum $\xi_1 + \xi_2$ by

$$\xi_1 + \xi_2 = E(\Delta_B, \nabla_A)(B \oplus B \rightarrowtail E_1 \oplus E_2 \twoheadrightarrow A \oplus A).$$

2.6. Show that if $\alpha_1, \alpha_2 : A' \to A$, then

$$(\alpha_1 + \alpha_2)^* = \alpha_1^* + \alpha_2^* : E(A,B) \to E(A',B),$$

using the addition given in Exercise 2.5. Deduce that $E(A,B)$ admits additive inverses (without using Theorem 2.4).

2.7. Show that the addition defined in Exercise 2.5 is commutative and associative (without using Theorem 2.4). [Thus $E(A,B)$ is an abelian group.]

2.8. Let $\mathbb{Z}_4 \rightarrowtail \mathbb{Z}_{16} \twoheadrightarrow \mathbb{Z}_4$ be the evident exact sequence. Construct its inverse in $E(\mathbb{Z}_4, \mathbb{Z}_4)$.

2.9. Show the group table of $E(\mathbb{Z}_8, \mathbb{Z}_{12})$.

3. Ext Using Injectives

Given two Λ-modules A, B, we defined in Section 2 a group $\operatorname{Ext}_\Lambda(A,B)$ by using a projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A:

$$\operatorname{Ext}_\Lambda(A,B) = \operatorname{coker}(\mu^* : \operatorname{Hom}_\Lambda(P,B) \to \operatorname{Hom}_\Lambda(R,B)).$$

Here we consider the dual procedure: Choose an *injective presentation* of B, i.e. an exact sequence $B \overset{\nu}{\rightarrowtail} I \overset{\eta}{\twoheadrightarrow} S$ with I injective, and define the group $\overline{\operatorname{Ext}}{}^\nu_\Lambda(A,B)$ as the cokernel of the map $\eta_* : \operatorname{Hom}_\Lambda(A,I) \to \operatorname{Hom}_\Lambda(A,S)$. Dualizing the proofs of Lemma 2.1, Corollary 2.2, and Theorem 2.3 one could show that $\overline{\operatorname{Ext}}{}^\nu_\Lambda(A,B)$ does not depend upon the chosen injective presentation, and that $\overline{\operatorname{Ext}}_\Lambda(-,-)$ can be made into a bifunctor, covariant in the second, contravariant in the first variable. Also, by dualizing the proof of Theorem 2.4 one proves that there is a natural equivalence of set-valued bifunctors between $E(-,-)$ and $\overline{\operatorname{Ext}}_\Lambda(-,-)$.

Here we want to give a different proof of the facts mentioned above which has the advantage of yielding yet another description of $E(-,-)$. In contrast to $\operatorname{Ext}_\Lambda(-,-)$ and $\overline{\operatorname{Ext}}_\Lambda(-,-)$, the new description will