Exercises:

3.1. Show that, if Λ is a principal ideal domain (p.i.d.), then an epimorphism $\beta: B \twoheadrightarrow B'$ induces an epimorphism $\beta_*: \operatorname{Ext}_\Lambda(A, B) \to \operatorname{Ext}_\Lambda(A, B')$. State and prove the dual.

3.2. Prove that $\operatorname{Ext}_{\mathbb{Z}}(A, \mathbb{Z}) \neq 0$ if A has elements of finite order.

3.3. Compute $\operatorname{Ext}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z})$, using an *injective* presentation of $\mathbb{Z}$.

3.4. Show that $\operatorname{Ext}_{\mathbb{Z}}(A, \operatorname{Ext}_{\mathbb{Z}}(B, C)) \cong \operatorname{Ext}_{\mathbb{Z}}(B, \operatorname{Ext}_{\mathbb{Z}}(A, C))$ when A, B, C are finitely-generated abelian groups.

3.5. Let the natural equivalences $\eta: E(-,-) \to \operatorname{Ext}_\Lambda(-,-)$ be defined by Theorem 2.4, $\sigma: \overline{\operatorname{Ext}}_\Lambda(-,-) \to \operatorname{Ext}_\Lambda(-,-)$ by Proposition 3.2, and

$$\bar{\eta}: E(-,-) \to \overline{\operatorname{Ext}}_\Lambda(-,-)$$

by dualizing the proof of Theorem 2.4. Show that $\sigma\ \eta = \bar{\eta}$.

4. Computation of some Ext-Groups

We start with the following

Lemma 4.1. (i) $\operatorname{Ext}_\Lambda\left(\bigoplus_i A_i, B\right) \cong \prod_i \operatorname{Ext}_\Lambda(A_i, B)$,

(ii) $\operatorname{Ext}_\Lambda\left(A, \prod_j B_j\right) \cong \prod_j \operatorname{Ext}_\Lambda(A, B_j)$.

Proof. We only prove assertion (i), leaving the other to the reader. For each i in the index set we choose a projective presentation $R_i \rightarrowtail P_i \twoheadrightarrow A_i$ of A_i. Then $\bigoplus_i R_i \rightarrowtail \bigoplus_i P_i \twoheadrightarrow \bigoplus_i A_i$ is a projective presentation of $\bigoplus_i A_i$. Using Proposition I.3.4 we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda\left(\bigoplus_i A_i, B\right) \arrow[r, tail] \arrow[d, "\wr"] & \operatorname{Hom}_\Lambda\left(\bigoplus_i P_i, B\right) \arrow[r] \arrow[d, "\wr"] & \operatorname{Hom}_\Lambda\left(\bigoplus_i R_i, B\right) \arrow[r, two heads] \arrow[d, "\wr"] & \operatorname{Ext}_\Lambda\left(\bigoplus_i A_i, B\right) \\
\prod_i \operatorname{Hom}_\Lambda(A_i, B) \arrow[r, tail] & \prod_i \operatorname{Hom}_\Lambda(P_i, B) \arrow[r] & \prod_i \operatorname{Hom}_\Lambda(R_i, B) \arrow[r, two heads] & \prod_i \operatorname{Ext}_\Lambda(A_i, B)
\end{tikzcd}$$

whence the result. $\square$

The reader may prefer to prove assertion (i) by using an injective presentation of B. Indeed in doing so it becomes clear that the two assertions of Lemma 4.1 are dual to each other.

In the remainder of this section we shall compute $\operatorname{Ext}_{\mathbb{Z}}(A, B)$ for A, B finitely-generated abelian groups. In view of Lemma 4.1 it is enough to consider the case where A, B are cyclic.

To facilitate the notation we shall write $\operatorname{Ext}(A,B)$ (for $\operatorname{Ext}_{\mathbb{Z}}(A,B)$) and $\operatorname{Hom}(A,B)$ (for $\operatorname{Hom}_{\mathbb{Z}}(A,B)$), whenever the groundring is the ring of integers.

Since $\mathbb{Z}$ is projective, one has

$$\operatorname{Ext}(\mathbb{Z},\mathbb{Z})=0=\operatorname{Ext}(\mathbb{Z},\mathbb{Z}_q)$$

by Proposition 2.6. To compute $\operatorname{Ext}(\mathbb{Z}_r,\mathbb{Z})$ and $\operatorname{Ext}(\mathbb{Z}_r,\mathbb{Z}_q)$ we use the projective presentation

$$\mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}_r$$

where μ is multiplication by r. We obtain the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{Z}_r,\mathbb{Z}) \arrow[r, tail] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, "\mu^*"] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r, two heads] \arrow[d, "\wr"] & \operatorname{Ext}(\mathbb{Z}_r,\mathbb{Z}) \\
0 \arrow[r, tail] & \mathbb{Z} \arrow[r, "\mu^*"] & \mathbb{Z} &
\end{tikzcd}$$

Since μ^* is again multiplication by r we obtain

$$\operatorname{Ext}(\mathbb{Z}_r,\mathbb{Z})\cong\mathbb{Z}_r\,.$$

Also the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{Z}_r,\mathbb{Z}_q) \arrow[r, tail] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}_q) \arrow[r, "\mu^*"] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}_q) \arrow[r, two heads] \arrow[d, "\wr"] & \operatorname{Ext}(\mathbb{Z}_r,\mathbb{Z}_q) \\
\mathbb{Z}_{(r,q)} \arrow[r, tail] & \mathbb{Z}_q \arrow[r, "\mu^*"] & \mathbb{Z}_q &
\end{tikzcd}$$

yields, since μ^* is multiplication by r,

$$\operatorname{Ext}(\mathbb{Z}_r,\mathbb{Z}_q)\cong\mathbb{Z}_{(r,q)}$$

where (r,q) denotes the greatest common divisor of r and q.

Exercises:

4.1. Show that there are p non-equivalent extensions $\mathbb{Z}_p \rightarrowtail E \twoheadrightarrow \mathbb{Z}_p$ for p a prime, but only two non-isomorphic groups E, namely $\mathbb{Z}_p\oplus\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$. How does this come about?

4.2. Classify the extension classes $[E]$, given by

$$\mathbb{Z}_m \rightarrowtail E \twoheadrightarrow \mathbb{Z}_n$$

under automorphisms of $\mathbb{Z}_m$ and $\mathbb{Z}_n$.

4.3. Show that if A is a finitely-generated abelian group such that $\operatorname{Ext}(A,\mathbb{Z})=0$, $\operatorname{Hom}(A,\mathbb{Z})=0$, then $A=0$.

4.4. Show that $\operatorname{Ext}(A,\mathbb{Z})\cong A$ if A is a finite abelian group.

4.5. Show that there is a natural equivalence of functors $\operatorname{Hom}(-,\mathbb{Q}/\mathbb{Z})\cong\operatorname{Ext}(-,\mathbb{Z})$ if both functors are restricted to the category of torsion abelian groups.

4.6. Show that extensions of finite abelian groups of relatively prime order split.

5. Two Exact Sequences

Here we shall deduce two exact sequences connecting Hom and Ext. We start with the following very useful lemma.

Lemma 5.1. *Let the following commutative diagram have exact rows.*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C & \longrightarrow 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' & \end{array}$$

Then there is a "connecting homomorphism" $\omega : \ker\gamma \to \operatorname{coker}\alpha$ *such that the following sequence is exact:*

$$\ker\alpha \xrightarrow{\mu_*} \ker\beta \xrightarrow{\varepsilon_*} \ker\gamma \xrightarrow{\omega} \operatorname{coker}\alpha \xrightarrow{\mu'_*} \operatorname{coker}\beta \xrightarrow{\varepsilon'_*} \operatorname{coker}\gamma\,. \quad (5.1)$$

If μ is monomorphic, so is μ_*: if ε' is epimorphic, so is ε'_*.

Proof. It is very easy to see – and we leave the verification to the reader – that the final sentence holds and that we have exact sequences

$$\ker\alpha \xrightarrow{\mu_*} \ker\beta \xrightarrow{\varepsilon_*} \ker\gamma\,,$$

$$\operatorname{coker}\alpha \xrightarrow{\mu'_*} \operatorname{coker}\beta \xrightarrow{\varepsilon'_*} \operatorname{coker}\gamma\,.$$

It therefore remains to show that there exists a homomorphism $\omega : \ker\gamma \to \operatorname{coker}\alpha$ "connecting" these two sequences. In fact, ω is defined as follows.

Let $c \in \ker\gamma$, choose $b \in B$ with $\varepsilon b = c$. Since $\varepsilon'\beta b = \gamma\varepsilon b = \gamma c = 0$ there exists $a' \in A'$ with $\beta b = \mu' a'$. Define $\omega(c) = [a']$, the coset of a' in $\operatorname{coker}\alpha$.

We show that ω is well defined, that is, that $\omega(c)$ is independent of the choice of b. Indeed, let $\bar{b} \in B$ with $\varepsilon\bar{b} = c$, then $\bar{b} = b + \mu a$ and

$$\beta(b + \mu a) = \beta b + \mu'\alpha a\,.$$

Hence $\bar{a}' = a' + \alpha a$, thus $[\bar{a}'] = [a']$. Clearly ω is a homomorphism.

Next we show exactness at $\ker\gamma$. If $c \in \ker\gamma$ is of the form εb for $b \in \ker\beta$, then $0 = \beta b = \mu' a'$, hence $a' = 0$ and $\omega(c) = 0$. Conversely, let $c \in \ker\gamma$ with $\omega(c) = 0$. Then $c = \varepsilon b$, $\beta b = \mu' a'$ and there exists $a \in A$ with $\alpha a = a'$. Consider $\bar{b} = b - \mu a$. Clearly $\varepsilon\bar{b} = c$, but

$$\beta\bar{b} = \beta b - \beta\mu a = \beta b - \mu' a' = 0\,,$$

hence $c \in \ker\gamma$ is of the form $\varepsilon\bar{b}$ with $\bar{b} \in \ker\beta$.

Finally we prove exactness at $\operatorname{coker}\alpha'$. Let $\omega(c) = [a'] \in \operatorname{coker}\alpha$. Thus $c = \varepsilon b$, $\beta b = \mu' a'$, and $\mu'_*[a'] = [\mu' a'] = [\beta b] = 0$. Conversely, let $[a'] \in \operatorname{coker}\alpha$ with $\mu'_*[a'] = 0$. Then $\mu' a' = \beta b$ for some $b \in B$ and $c = \varepsilon b \in \ker\gamma$. Thus $[a'] = \omega(c)$. $\square$

For an elegant proof of Lemma 5.1 using Lemma 3.1, see Exercise 5.1.

We remark that the sequence (5.1) is natural in the obvious sense: If we are given a commutative diagram with exact rows

$$
\begin{tikzcd}
 & D \arrow[rr] \arrow[dd] & & E \arrow[rr] \arrow[dd] & & F \arrow[rr] \arrow[dd] & & 0 & \\
 & & A \arrow[ul] \arrow[rr, crossing over] \arrow[dd] & & B \arrow[ul] \arrow[rr, crossing over] \arrow[dd] & & C \arrow[ul] \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[r] & D' \arrow[rr] & & E' \arrow[rr] & & F' & & & \\
 & 0 \arrow[r] & A' \arrow[ul] \arrow[rr] & & B' \arrow[ul] \arrow[rr] & & C' \arrow[ul] & &
\end{tikzcd}
\tag{5.2}
$$

we obtain a mapping from the sequence stemming from the front diagram to the sequence stemming from the back diagram.

We use Lemma 5.1 to prove

Theorem 5.2. *Let A be a Λ-module and let $B' \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} B''$ be an exact sequence of Λ-modules. There exists a "connecting homomorphism" $\omega: \mathrm{Hom}_\Lambda(A, B'') \to \mathrm{Ext}_\Lambda(A, B')$ such that the following sequence is exact and natural*

$$
\begin{aligned}
0 \to \mathrm{Hom}_\Lambda(A, B') &\xrightarrow{\varphi_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\psi_*} \mathrm{Hom}_\Lambda(A, B'') \\
&\xrightarrow{\omega} \mathrm{Ext}_\Lambda(A, B') \xrightarrow{\varphi_*} \mathrm{Ext}_\Lambda(A, B) \xrightarrow{\psi_*} \mathrm{Ext}_\Lambda(A, B'') .
\end{aligned}
\tag{5.3}
$$

This sequence is called the *Hom-Ext-sequence (in the second variable).*

Proof. Choose any projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A and consider the following diagram with exact rows and columns

$$
\begin{tikzcd}
 & \mathrm{Hom}_\Lambda(A, B') \arrow[r, dotted, "\varphi_*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, B) \arrow[r, dotted, "\psi_*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, B'') \arrow[d, tail] & \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(P, B') \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(R, B') \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R, B) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R, B'') \arrow[d, two heads] & \\
 & \mathrm{Ext}_\Lambda(A, B') \arrow[r, dotted, "\varphi_*"] & \mathrm{Ext}_\Lambda(A, B) \arrow[r, dotted, "\psi_*"] & \mathrm{Ext}_\Lambda(A, B'') &
\end{tikzcd}
\tag{5.4}
$$

The second and third rows are exact by Theorem 1.2.1. In the second row $\psi_*: \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(P, B'')$ is epimorphic since P is projective (Theorem I.4.7). Applying Lemma 5.1 to the two middle rows of the diagram we obtain the homomorphism ω and the exactness of the resulting sequence.

Let $\alpha: A' \to A$ be a homomorphism and let $R' \rightarrowtail P' \twoheadrightarrow A'$ be a projective presentation of A'. Choose $\pi: P' \to P$ and $\sigma: R' \to R$ such that the diagram

$$\begin{tikzcd} R' \arrow[r, tail] \arrow[d, "\sigma"] & P' \arrow[r, two heads] \arrow[d, "\pi"] & A' \arrow[d, "\alpha"] \\ R \arrow[r, tail] & P \arrow[r, two heads] & A \end{tikzcd}$$

is commutative. Then α, π, σ induce a mapping from diagram (5.4) associated with $R \rightarrowtail P \twoheadrightarrow A$ to the corresponding diagram associated with $R' \rightarrowtail P' \twoheadrightarrow A'$. The two middle rows of these diagrams form a diagram of the kind (5.2). Hence the Hom-Ext sequence corresponding to A is mapped into the Hom-Ext sequence corresponding to A'. In particular – choosing $\alpha = 1_A : A \to A$ – this shows that ω is independent of the chosen projective presentation.

Analogously one proves that homomorphisms β', β, β'' which make the diagram

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d, "\beta'"] & B \arrow[r, two heads] \arrow[d, "\beta"] & B'' \arrow[d, "\beta''"] \\ C' \arrow[r, tail] & C \arrow[r, two heads] & C'' \end{tikzcd}$$

commutative induce a mapping from the Hom-Ext sequence associated with the short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ to the Hom-Ext sequence associated with the short exact sequence $C' \rightarrowtail C \twoheadrightarrow C''$. In particular the following square is commutative.

$$\begin{tikzcd} \mathrm{Hom}_\Lambda(A, B'') \arrow[r, "\omega"] \arrow[d, "\beta''_*"] & \mathrm{Ext}_\Lambda(A, B') \arrow[d, "\beta'_*"] \\ \mathrm{Hom}_\Lambda(A, C'') \arrow[r, "\omega"] & \mathrm{Ext}_\Lambda(A, C') \end{tikzcd}$$

This completes the proof of Theorem 5.2. □

We make the following remark with respect to the connecting homomorphism $\omega: \mathrm{Hom}_\Lambda(A, B'') \to \mathrm{Ext}_\Lambda(A, B')$ as constructed in the proof of Theorem 5.2. Given $\alpha: A \to B''$ we define maps π, σ such that the diagram

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, "\sigma"] & P \arrow[r, two heads] \arrow[d, "\pi"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail, "\varphi"] & B \arrow[r, two heads, "\psi"] & B'' \end{tikzcd}$$

is commutative. The construction of ω in diagram (5.4) shows that $\omega(\alpha) = [\sigma] \in \mathrm{Ext}_\Lambda(A, B')$. Now let E be the pull-back of (ψ, α). We then have a map $\pi' : P \to E$ such that the diagram

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[d, "\sigma"] & P \arrow[r, two heads] \arrow[d, "\pi"] & A \arrow[d, equal] \\
B' \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d] & A \arrow[d, "\alpha"] \\
B' \arrow[r, tail] & B \arrow[r, two heads] & B''
\end{tikzcd}$$

is commutative. By the definition of the equivalence

$$\xi : \mathrm{Ext}_\Lambda(A, B') \xrightarrow{\sim} E(A, B')$$

in Theorem 2.4 the element $\xi[\sigma]$ is represented by the extension $B' \rightarrowtail E \twoheadrightarrow A$.

We now introduce a *Hom-Ext-sequence in the first variable.*

Theorem 5.3. *Let B be a Λ-module and let $A' \overset{\varphi}{\rightarrowtail} A \overset{\psi}{\twoheadrightarrow} A''$ be a short exact sequence. Then there exists a connecting homomorphism*

$$\omega : \mathrm{Hom}_\Lambda(A', B) \to \mathrm{Ext}_\Lambda(A'', B)$$

such that the following sequence is exact and natural

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\Lambda(A'', B) \xrightarrow{\psi^*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varphi^*} \mathrm{Hom}_\Lambda(A', B) \\ \xrightarrow{\omega} \mathrm{Ext}_\Lambda(A'', B) \xrightarrow{\psi^*} \mathrm{Ext}_\Lambda(A, B) \xrightarrow{\varphi^*} \mathrm{Ext}_\Lambda(A', B)\,. \end{aligned} \tag{5.5}$$

The reader notes that, if Ext is identified with $\overline{\mathrm{Ext}}$, Theorem 5.3 becomes the dual of Theorem 5.2 and that it may be proved by proceeding dually to Theorem 5.2 (see Exercises 5.4, 5.5). We prefer, however, to give a further proof using only projectives and thus avoiding the use of injectives. For our proof we need the following lemma, which will be invoked again in Chapter IV.

Lemma 5.4. *To a short exact sequence $A' \overset{\varphi}{\rightarrowtail} A \overset{\psi}{\twoheadrightarrow} A''$ and to projective presentations $\varepsilon' : P' \twoheadrightarrow A'$ and $\varepsilon'' : P'' \twoheadrightarrow A''$ there exists a projective presentation $\varepsilon : P \twoheadrightarrow A$ and homomorphisms $\iota : P' \to P$ and $\pi : P \to P''$ such that the following diagram is commutative with exact rows*

$$\begin{tikzcd}
P' \arrow[r, tail, "\iota"] \arrow[d, "\varepsilon'"] & P \arrow[r, two heads, "\pi"] \arrow[d, "\varepsilon"] & P'' \arrow[d, "\varepsilon''"] \\
A' \arrow[r, tail, "\varphi"] & A \arrow[r, two heads, "\psi"] & A''
\end{tikzcd}$$

Proof. Let $P = P' \oplus P''$, let $\iota : P' \to P' \oplus P''$ be the canonical injection, $\pi : P' \oplus P'' \to P''$ the canonical projection. We define ε by giving the components. The first component is $\varphi \varepsilon' : P' \to A$; for the second we use the fact that P'' is projective to construct a map $\chi : P'' \to A$ which makes the triangle

$$\begin{tikzcd} & P'' \arrow[dl, "\chi"'] \arrow[d, "\varepsilon''"] \\ A \arrow[r, two heads, "\psi"'] & A'' \end{tikzcd}$$

commutative, and take χ as the second component of ε. It is plain that with this definition the above diagram commutes. By Lemma I.1.1 ε is epimorphic. ☐

Proof of Theorem 5.3. Using Lemma 5.4 projective presentations may be chosen such that the following diagram is commutative with short exact middle row

$$\begin{tikzcd} R' \arrow[r] \arrow[d, tail] & R \arrow[r] \arrow[d, tail] & R'' \arrow[d, tail] \\ P' \arrow[r, tail, "\iota"] \arrow[d, two heads, "\varepsilon'"] & P \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\varepsilon"] & P'' \arrow[d, two heads, "\varepsilon''"] \\ A' \arrow[r, tail, "\varphi"] & A \arrow[r, two heads, "\psi"] & A'' \end{tikzcd}$$

By Lemma 5.1 applied to the second and third row the top row is short exact, also. Applying $\mathrm{Hom}_\Lambda(-, B)$ we obtain the following diagram

$$\begin{tikzcd} & \mathrm{Hom}_\Lambda(A'', B) \arrow[r, dotted, "\psi^*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, B) \arrow[r, dotted, "\varphi^*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A', B) \arrow[d, tail] & \\ 0 \arrow[r] & \mathrm{Hom}_\Lambda(P'', B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P. B) \arrow[r, "\iota^*"] \arrow[d] & \mathrm{Hom}_\Lambda(P', B) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_\Lambda(R'', B) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R, B) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R', B) \arrow[d, two heads] & \\ & \mathrm{Ext}_\Lambda(A'', B) \arrow[r, dotted, "\psi^*"] & \mathrm{Ext}_\Lambda(A, B) \arrow[r, dotted, "\varphi^*"] & \mathrm{Ext}_\Lambda(A', \mathrm{B}) & \end{tikzcd} \tag{5.6}$$

By Theorem 1.2.2 the second and third rows are exact. In the second row $\iota^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(P', B)$ is epimorphic since $P = P' \oplus P''$, so $\mathrm{Hom}_\Lambda(P' \oplus P'', B) \cong \mathrm{Hom}_\Lambda(P', B) \oplus \mathrm{Hom}_\Lambda(P'', B)$. Lemma 5.1 now yields the Hom-Ext sequence claimed. As in the proof of Theorem 5.2 one shows that ω is independent of the chosen projective presentations. Also, one proves that the Hom-Ext sequence in the first variable is natural with respect to homomorphisms $\beta : B \to B'$ and with respect to maps γ', γ, γ'' making the diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, "\gamma'"] & A \arrow[r, two heads] \arrow[d, "\gamma"] & A'' \arrow[d, "\gamma''"] \\ C' \arrow[r, tail] & C \arrow[r, two heads] & C'' \end{tikzcd}$$

commutative. ☐

If we try to describe the connecting homomorphism

$$\omega: \mathrm{Hom}_\Lambda(A', B) \to \mathrm{Ext}_\Lambda(A'', B)$$

in terms of extensions, it is natural to consider the push-out E of $\alpha: A' \to B$ and $\varphi: A' \to A$ and to construct the diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, "\alpha"] & A \arrow[r, two heads] \arrow[d] & A'' \arrow[d, equal] \\ B \arrow[r, tail] & E \arrow[r, two heads] & A'' \end{tikzcd}$$

We then consider the presentation $R'' \rightarrowtail P'' \twoheadrightarrow A''$ and note that the map $\chi: P'' \to A$ constructed in the proof of Lemma 5.4 induces σ such that the diagram

$$\begin{tikzcd} R'' \arrow[r, tail] \arrow[d, "\sigma"] & P'' \arrow[r, two heads] \arrow[d, "\chi"] & A'' \arrow[d, equal] \\ A' \arrow[r, tail] \arrow[d, "\alpha"] & A \arrow[r, two heads] \arrow[d] & A'' \arrow[d, equal] \\ B \arrow[r, tail] & E \arrow[r, two heads] & A'' \end{tikzcd}$$

is commutative. Now the definition of $\omega(\alpha)$ in diagram (5.6) is via the map $\varrho: R'' \to B$ which is obtained as $\varrho = \alpha\tau$ in

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, two heads] & P' \oplus P'' \arrow[r, "\pi_1"] & P' \arrow[r, "\varepsilon'"] & A' \arrow[r, "\alpha"] & B \\ R'' \arrow[urrr, "\tau"'] \end{tikzcd}$$

But by the definition of $\varepsilon: P' \oplus P'' \to A$ in Lemma 5.4, the sum of the two maps

$$R \twoheadrightarrow R'' \xrightarrow{\sigma} A' \overset{\varphi}{\rightarrowtail} A$$

$$R \twoheadrightarrow R'' \xrightarrow{\tau} A' \overset{\varphi}{\rightarrowtail} A$$

is zero. Hence $\sigma = -\tau$, so that the element $-\omega(\alpha) = [-\tau]$ is represented by the extension $B \rightarrowtail E \twoheadrightarrow A''$.

Corollary 5.5. *The Λ-module A is projective if and only if* $\mathrm{Ext}_\Lambda(A, B) = 0$ *for all Λ-modules B.*

Proof. Suppose A is projective. Then $1: A \xrightarrow{\sim} A$ is a projective presentation, whence $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. Conversely, suppose $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. Then for any short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ the sequence

$$0 \to \mathrm{Hom}_\Lambda(A, B') \to \mathrm{Hom}_\Lambda(A, B) \to \mathrm{Hom}_\Lambda(A, B'') \to 0$$

is exact. By Theorem I.4.7 A is projective. $\square$

The reader may now easily prove the dual assertion.

Corollary 5.6. *The Λ-module B is injective if and only if* $\mathrm{Ext}_\Lambda(A, B) = 0$ *for all Λ-modules A.* □

In the special case where Λ is a principal ideal domain we obtain

Corollary 5.7. *Let Λ be a principal ideal domain. Then the homomorphisms* $\psi_* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A, B'')$ *in sequence* (5.3) *and*

$$\varphi^* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A', B)$$

in sequence (5.5) *are epimorphic.*

Proof. Over a principal ideal domain Λ submodules of projective modules are projective. Hence in diagram (5.4) R is projective; thus

$$\psi_* : \mathrm{Hom}_\Lambda(R, B) \to \mathrm{Hom}_\Lambda(R, B'')$$

is epimorphic, and hence $\psi_* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A, B'')$ is epimorphic. In diagram (5.6), R'' is projective. Hence the short exact sequence $R' \rightarrowtail R \twoheadrightarrow R''$ splits and it follows that $\varphi^* : \mathrm{Hom}_\Lambda(R, B) \to \mathrm{Hom}_\Lambda(R', B)$ is epimorphic. Hence $\varphi^* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A', B)$ is epimorphic. □

We remark, that if Λ is *not* a principal ideal domain the assertions of Corollary 5.7 are false in general (Exercise 5.3).

Exercises:

5.1. Consider the following diagram

$$\begin{tikzcd}
 & \ker\beta \arrow[r, "\varepsilon_*"] \arrow[d] \arrow[dr, phantom, "\Sigma_1"] & \ker\gamma \arrow[d] & \\
A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu'"] \arrow[d] \arrow[dr, phantom, "\Sigma_4"] & B' \arrow[r, "\varepsilon'"] \arrow[d] & C' \\
 & \operatorname{coker}\alpha \arrow[r, "\mu'_*"] & \operatorname{coker}\beta &
\end{tikzcd}$$

with all sequences exact. Show that with the terminology of Lemma 3.1 we have $\mathrm{Im}\,\Sigma_1 \cong \operatorname{coker}\varepsilon_*$, $\mathrm{Ker}\,\Sigma_4 \cong \ker\mu'_*$. Show $\mathrm{Im}\,\Sigma_1 \cong \mathrm{Ker}\,\Sigma_4$ by a repeated application of Lemma 3.1. With that result prove Lemma 5.1.

5.2. Given $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ (not necessarily exact) deduce from Lemma 5.1 (or prove otherwise) that there is a natural exact sequence

$$0 \to \ker\alpha \to \ker\beta\alpha \to \ker\beta \to \operatorname{coker}\alpha \to \operatorname{coker}\beta\alpha \to \operatorname{coker}\beta \to 0\,.$$

5.3. Show that if R is not projective there exists a module B with $\mathrm{Ext}_\Lambda(R, B) \neq 0$. Suppose that in the projective presentation $R \overset{\varphi}{\rightarrowtail} P \overset{\psi}{\twoheadrightarrow} A$ of A the module R is *not* projective. Deduce that $\varphi^* : \mathrm{Ext}_\Lambda(P, B) \to \mathrm{Ext}_\Lambda(R, B)$ is not epimorphic. Compare with Corollary 5.7.

5.4. Prove Theorem 5.3 by using the definition of Ext by injectives and interpret the connecting homomorphism in terms of extensions. Does one get the same connecting homomorphism as in our proof of Theorem 5.3?

5.5. Prove Theorem 5.2 using the definition of Ext by injectives. (Use the dual of Lemma 5.4.) Does one get the same connecting homomorphism as in our proof of Theorem 5.2?

5.6. Establish equivalences of Ext_Λ and $\overline{\mathrm{Ext}}_\Lambda$ using (i) Theorem 5.2, (ii) Theorem 5.3. Does one get the same equivalences?

5.7. Evaluate the groups and homomorphisms in the appropriate sequences (of Theorems 5.2, 5.3) when

(i) $0 \to A' \to A \to A'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$, B is $\mathbb{Z}_4$;
(ii) $0 \to A' \to A \to A'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$, B is $\mathbb{Z}_4$;
(iii) A is $\mathbb{Z}_4$, $0 \to B' \to B \to B'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$;
(iv) A is $\mathbb{Z}_4$, $0 \to B' \to B \to B'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$;
(v) $0 \to A' \to A \to A'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$, B is $\mathbb{Z}$.

5.8. For any abelian group A, let

$$mA = \{b \in A \mid b = ma, a \in A\},$$
$$_mA = \{a \in A \mid ma = 0\},$$
$$A_m = A/mA.$$

Show that there are exact sequences

$$0 \to \mathrm{Ext}(mA, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(_mA, \mathbb{Z}) \to 0,$$

$$0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Hom}(mA, \mathbb{Z}) \to \mathrm{Ext}(A_m, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(mA. \mathbb{Z}) \to 0.$$

and that $\mathrm{Hom}(A, \mathbb{Z}) \cong \mathrm{Hom}(mA, \mathbb{Z})$.
Prove the following assertions:
(i) $_mA = 0$ if and only if $\mathrm{Ext}(A, \mathbb{Z})_m = 0$;
(ii) if $A_m = 0$ then $_m\mathrm{Ext}(A. \mathbb{Z}) = 0$;
(iii) if $_m\mathrm{Ext}(A, \mathbb{Z}) = 0 = \mathrm{Hom}(A, \mathbb{Z})$, then $A_m = 0$.
Give a counterexample to show that the converse of (ii) is not true.
(Hint: an abelian group B such that $mB = 0$ is a direct sum of cyclic groups.)

6. A Theorem of Stein-Serre for Abelian Groups

By Corollary 5.5 A is projective if and only if $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. The question naturally arises as to whether it is necessary to use *all* Λ-modules B in $\mathrm{Ext}_\Lambda(A, B)$ to test whether A is projective; might it not happen that there exists a small family of Λ-modules B_i such that if $\mathrm{Ext}_\Lambda(A, B_i) = 0$ for every B_i in the family, then A is projective? Of course, as is easily shown, A is projective if $\mathrm{Ext}_\Lambda(A, R) = 0$ where $R \rightarrowtail P \twoheadrightarrow A$ is a projective presentation of A, but our intention is that the family B_i may be chosen independently of A.

For $\Lambda = \mathbb{Z}$ and A finitely-generated there is a very simple criterion for A to be projective (i.e. free): *If A is a finitely generated abelian group,*