24/7/16

MAST90068 – Lecture 2

Last time we defined categories, pullbacks and argued that the fiber product in Top is an example of a pullback. We begin today by proving two statements from last lecture. But first some definitions.

Defⁿ Let $\mathcal{C}$ be a category and $f : A \longrightarrow B$ a morphism. We say

- f is a monomorphism if for every object C and pair of morphisms $u, v : C \longrightarrow A$, $fu = fv$ implies $u = v$.
- f is an epimorphism if for every object C and pair of morphisms $u, v : B \longrightarrow C$, $uf = vf$ implies $u = v$.
- f is an isomorphism if there exists $f' : B \longrightarrow A$ with both $ff' = 1_B$ and $f'f = 1_A$.

Ex 1 f is an isomorphism $\not\Leftrightarrow$ f is mono & epi (give a counterexample)

Defⁿ
- Set is the category of sets and functions
- Ab is the category of abelian groups and group homomorphisms.
- Rng is the category of (not necessarily commutative) rings with unit, and ring homomorphisms (which preserve the unit).

Ex 2 Verify these are categories (and maybe fret about "the set of all sets")

Ex 3 In Set, Ab mono = injective, epi = surjective, iso = bijective

Ex 4 If R is a ring and $S \subseteq R$ a multiplicatively closed set, $R \longrightarrow S^{-1}R$ is an epimorphism in the cat. of commutative rings, but is not surjective.

Lemma Given a diagram in a category $\mathcal{C}$

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y \arrow[r, "\theta"'] & Z \end{tikzcd} \tag{2.1}$$

and two pullbacks (U, p_X, p_Y) and (U', p_X', p_Y') there is a unique morphism $\psi : U' \longrightarrow U$ such that $p_X \psi = p_X'$ and $p_Y \psi = p_Y'$, and this ψ is an isomorphism.

Note We say "the pullback is unique up to unique isomorphism". More generally, as we will learn, all limits and colimits are unique in this sense.

Proof Consider

$$\begin{tikzcd} U' \arrow[dr, dashed, bend right, "\psi"'] \arrow[drr, bend left, "p_X'"] \arrow[ddr, bend right, "p_Y'"'] & & \\ & U \arrow[ul, dashed, bend right, "\psi'"'] \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "\pi"] \\ & Y \arrow[r, "\theta"'] & Z \end{tikzcd} \tag{2.2}$$

By the universal property of (U, p_X, p_Y) there exists ψ with $p_X \psi = p_X'$ and $p_Y \psi = p_Y'$ (and this ψ is unique). Similarly by the universal property of (U', p_X', p_Y') there exists unique ψ' with $p_X' \psi' = p_X$, $p_Y' \psi' = p_Y$.

It remains to argue that $\psi \psi' = 1_U$, $\psi' \psi = 1_{U'}$. But

$$p_X \psi \psi' = p_X' \psi' = p_X = p_X 1_U, \quad p_Y \psi \psi' = p_Y' \psi' = p_Y = p_Y 1_U$$

so by the uniqueness in the def^{N} of pullback, $\psi \psi' = 1_U$, similarly $\psi' \psi = 1_{U'}$. $\square$

applied to the pair (p_X, p_Y)

<u>Theorem</u> Let $\mathcal{C}$ be a category and consider a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & Z \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd} \tag{3.1}$$

in which II is a pullback. Then I is a pullback if and only if the outer square (X, Z, A, C) is a pullback.

<u>Proof</u> Suppose I, II are both pullbacks, and let morphisms $T \longrightarrow A$, $T \longrightarrow Z$ be given such that (with the obvious meaning)

$$T \longrightarrow A \longrightarrow C = T \longrightarrow Z \longrightarrow C.$$

Apply the pullback II to the pair $(T \to A \to B, T \to Z)$ to produce unique $T \longrightarrow Y$ with $T \to Y \to B = T \to A \to B$ and $T \to Y \to Z = T \to Z$. Then apply pullback I to this $T \to Y$ and $T \to A$ to produce $T \to X$ with $T \to X \to A = T \to X \to Y$. By construction also $T \to X \to Y \to Z = T \to Z$ so for the forward implication of the theorem, we need only show $T \to X$ is <u>unique</u> (with $T \to X \to A = T \to A$ and $T \to X \to Y \to Z = T \to Z$).

Suppose $T \xrightarrow{\bullet} X$ to be a morphism with these properties. Using pullback II we deduce $T \xrightarrow{\bullet} X \longrightarrow Y = T \longrightarrow Y$, and then from pullback I we conclude that $T \xrightarrow{\bullet} X = T \longrightarrow X$, as claimed. $\square$

<u>Ex 5</u> Finish the proof by showing that if II & the outer square are pullbacks then so is I.

<u>Ex 6</u> The fiber product $X \times_Z Y$ gives the pullback in <u>Ab</u>.

<u>Ex 7</u> Let $\varphi : A \longrightarrow B$ be a morphism of abelian groups and $X \subseteq B$ a subgroup. Then the following diagram is a pullback

$$\begin{tikzcd} \varphi^{-1}X \arrow[r, dashed, "\varphi'"] \arrow[d, hook] & X \arrow[d, hook] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd} \qquad (\text{where } \varphi'(x) = \varphi(x)). \qquad (4.1)$$

In particular for $X = 0$, $\varphi^{-1}X = \operatorname{Ker}\varphi$ is a pullback.

The core "philosophy" of category theory is that <u>morphisms</u> (i.e. transformations) are the central concept of mathematics, not <u>objects</u>. This point of view is not really original to category theory, it appears famously in Klein's Erlangen program (which proposed to do geometry via transformation groups) and arguably goes back to the Greeks. But category theory is a powerful technical tool for making this philosophy manifest.

So, obviously we should now talk about <u>morphisms between categories</u>! These are called <u>functors</u>. By the above philosophy, category theory is really "functor theory".

<u>Def^N</u> Let $\mathcal{C}, \mathcal{D}$ be categories. A <u>functor</u> $F : \mathcal{C} \longrightarrow \mathcal{D}$ is the data of

(1) A function $F : ob(\mathcal{C}) \longrightarrow ob(\mathcal{D})$, written $A \longmapsto FA$.

(2) For each pair of objects A, B of $\mathcal{C}$ a function

$$F_{AB} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA, FB)$$

written $f \longmapsto F(f)$,

subject to the following axioms

(3) For every object A of $\mathcal{C}$, $F(1_A) = 1_{F(A)}$.

(4) For every triple of objects A, B, C the diagram

$$\begin{tikzcd}
\mathcal{C}(B,C) \times \mathcal{C}(A,B) \arrow[r] \arrow[d, "F \times F"] & \mathcal{C}(A,C) \arrow[d, "F"] \\
\mathcal{D}(FB,FC) \times \mathcal{D}(FA,FB) \arrow[r] & \mathcal{D}(FA,FC)
\end{tikzcd}$$

commutes, i.e. for $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

<u>Note</u> This is sometimes called a <u>covariant</u> functor. We will see <u>contravariant</u> functors shortly.

<u>Example</u> $id : \mathcal{C} \to \mathcal{C}$, $id(A) = A$, $id(f) = f$ is a functor, the <u>identity</u> functor.

Ex 8 If $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $G: \mathcal{C}_2 \longrightarrow \mathcal{C}_3$ are functors, then so is $G \circ F: \mathcal{C}_1 \longrightarrow \mathcal{C}_3$ defined on objects by

$$(G \circ F)(A) = G(F(A))$$

and on morphisms by

$$(G \circ F)(f) = G(F(f)).$$

Note If $H: \mathcal{C}_3 \longrightarrow \mathcal{C}_4$ is a third functor then it is clear that

$$H \circ (G \circ F) = (H \circ G) \circ F,$$

and that if $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor then $\mathrm{id}_{\mathcal{D}} \circ F = F = F \circ \mathrm{id}_{\mathcal{C}}$. So there seems to be a category $\underline{Cat}$ whose objects are categories and whose morphisms are functors. However, we have to be careful with set-theoretic issues (later).

Example (a) The <u>forgetful functor</u> $F: \underline{Ab} \longrightarrow \underline{Set}$ sends an abelian group to its underlying set, and a homomorphism to itself, i.e.

$$F(A,+) = A, \quad F(f) = f. \tag{6.1}$$

(b) The <u>free group</u> functor $V: \underline{Set} \longrightarrow \underline{Ab}$ sends a set to the free abelian group on that set, i.e.

$$V(S) = \{ f: S \longrightarrow \mathbb{Z} \mid f \text{ has finite support} \} \tag{6.2}$$

with operation $(f+g)(s) = f(s) + g(s)$.

$\operatorname{supp}(f) = \{ s \in S \mid f(s) \neq 0 \}$

The functor is defined on morphisms by, for $t : S \to S'$

$$V(t) : V(S) \longrightarrow V(S')$$

$$V(t)(f)(s') = \sum_{s \in t^{-1}(s')} f(s) \qquad (7.1)$$

$$\begin{tikzcd} S \arrow[r, "t"] \arrow[dr, "f"'] & S' \ni s' \\ & \mathbb{Z} \end{tikzcd}$$

This is well-defined (i.e. the sum is over a possibly infinite set, but only finitely many summands can be nonzero).

<u>Ex 9</u> (a) $V(S)$ is an abelian group.
(b) For all $f \in V(S)$, $V(t)(f)$ has finite support and therefore lies in $V(S')$.
(c) $V(t)$ is a morphism of abelian groups.
(d) V is a functor.

<u>Note</u> Observe that there is a natural embedding (a function between sets which is injective) $\iota_S : S \longrightarrow V(S)$ given by delta functions

$$\iota_S(x)(s) = \begin{cases} 1 & \text{if } s = x \\ 0 & \text{else} \end{cases} \qquad (7.2)$$

Identifying S with the image of this embedding (i.e. writing x for $\iota_S(x)$) and using that any abelian group is a $\mathbb{Z}$-module, we may write any element f of $V(S)$ uniquely as a linear combination

$$f = \sum_{s \in S} f(s) \cdot s \qquad f_s \in \mathbb{Z} \qquad (7.3)$$

We call $V(S)$ the <u>free abelian group</u> on the set S.

<u>Note</u> In a category $\mathscr{C}$ we often write $\mathrm{Hom}_{\mathscr{C}}(A,B)$ for $\mathscr{C}(A,B)$.

Using category theory we can give a precise meaning to this word "free".

<u>Lemma</u> For any set S and abelian group B there is a bijection

$$\Phi : \mathrm{Hom}_{\underline{Ab}}(V(S), B) \longrightarrow \mathrm{Hom}_{\underline{Set}}(S, F(B))$$

$$\Phi(\varphi) = \varphi|_S \qquad (8.1)$$

where $F : \underline{Ab} \longrightarrow \underline{Set}$ is the forgetful functor, and $\varphi|_S$ means restrict φ to S, understood as a subset of $V(S)$ via the embedding ι_S.

<u>Proof</u> Φ is clearly well-defined.

- <u>Φ is injective</u> if $\Phi(\varphi) = \Phi(\varphi')$ then φ and φ' agree on $S \subseteq V(S)$, so by linearity they agree, i.e. for $f \in V(S)$

$$\begin{aligned}
\varphi(f) &= \varphi\big(\textstyle\sum_s f(s)\cdot s\big) \\
&= \textstyle\sum_s \varphi(f(s)\cdot s) && (f(s) \in \mathbb{Z} \text{ recall}) \\
&= \textstyle\sum_s f(s)\cdot \varphi(s) && (\text{meaning } \varphi(\iota_S(s))) \\
&= \textstyle\sum_s f(s)\cdot \varphi'(s) \\
&= \varphi'\big(\textstyle\sum_s f(s)\cdot s\big) \\
&= \varphi'(f).
\end{aligned}$$

Hence $\varphi = \varphi'$ and Φ is injective.

• <u>Φ is surjective</u> let $h: S \longrightarrow B$ be a function, and define

$$\begin{aligned} \varphi_h : V(S) &\longrightarrow B \\ \varphi_h(f) &= \sum_{s \in S} f(s) \cdot h(s) \end{aligned} \qquad (9.1)$$

This is clearly linear in f, and well-defined, and restricts to h on S, since $\varphi_h(s) = \sum_{s' \in S} \delta_{ss'} h(s) = h(s)$, so $\Phi(\varphi_h) = h$, as required. $\square$

<u>Ex 10</u> Check that Φ is <u>natural</u> in S, B. That is, for a function $\alpha: S \to S'$ and homomorphism of groups $\beta: B \to B'$ show that the diagrams below both commute

$$\begin{tikzcd} \mathrm{Hom}_{\underline{Ab}}(V(S), B) \arrow[r, "\Phi_{S,B}"] \arrow[d, "\varphi \mapsto \beta \circ \varphi"] & \mathrm{Hom}_{\underline{Set}}(S, F(B)) \arrow[d, "h \mapsto F(\beta) \circ h"] \\ \mathrm{Hom}_{\underline{Ab}}(V(S), B') \arrow[r, "\Phi_{S,B'}"'] & \mathrm{Hom}_{\underline{Set}}(S, F(B')) \end{tikzcd} \qquad (9.2)$$

$$\begin{tikzcd} \mathrm{Hom}_{\underline{Ab}}(V(S), B) \arrow[r, "\Phi_{S,B}"] & \mathrm{Hom}_{\underline{Set}}(S, F(B)) \\ \mathrm{Hom}_{\underline{Ab}}(V(S'), B) \arrow[r, "\Phi_{S',B}"'] \arrow[u, "\varphi \mapsto \varphi \circ V(\alpha)"] & \mathrm{Hom}_{\underline{Set}}(S', F(B)) \arrow[u, "h \mapsto h \circ \alpha"'] \end{tikzcd} \qquad (9.3)$$

(The isomorphism Φ together with naturality (9.2), (9.3) says that V is <u>left adjoint</u> to F. In general, in category theory "free" always means "left adjoint to a forgetful functor").

Ex 11 Consider the forgetful functor $F: \underline{\mathrm{Rng}} \longrightarrow \underline{\mathrm{Mon}}$ where $\underline{\mathrm{Mon}}$ denotes the category of all monoids and monoid homomorphisms, and

$$F(R,+,\cdot) = (R,\cdot)$$

note, the underlying multiplicative monoid

sends a ring to its underlying monoid. Using the above for inspiration, construct a functor $V: \underline{\mathrm{Mon}} \longrightarrow \underline{\mathrm{Rng}}$ together with a natural bijection for each monoid G and ring R

$$\Phi : \mathrm{Hom}_{\underline{\mathrm{Rng}}}(V(G), R) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathrm{Mon}}}(G, F(R)).$$

(Optional) What is the relation between $V(G)$-modules and $\mathbb{Z}$-linear representations of G?