Lecture 20

We have now seen a description of injective Λ-modules as precisely the divisible modules for any PID (so e.g. $\mathbb{Z}$ or $k[x]$, k a field). Next we use this to show that the category of Λ-modules has "enough" injectives for any ring Λ. This is essential for us to continue developing homological algebra beyond Ext^1. The discussion will be phrased in terms of injective cogenerators which are a dual notion to projective generators (Λ is a projective generator in Λ-Mod). However, we should be careful not to think of injectives as "cofree" in a naive sense:

Ex 1 A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ with a left (right) adjoint preserves limits (colimits)

Ex 2 The forgetful functor $F : \Lambda\text{-Mod} \longrightarrow \underline{\mathrm{Set}}$ for a ring Λ has a left adjoint but no right adjoint (there are no "cofree" modules).

Let $\mathcal{C}$ be a category with all products and coproducts (i.e. these limits & colimits exist).

Def^n We say $\mathcal{C}$ has enough projectives if for every object X there exists an epimorphism $P \longrightarrow X$ with P projective. Dually $\mathcal{C}$ has enough injectives if for every X there is a monomorphism $X \longrightarrow I$ with I injective.

Clearly Λ-Mod has enough projectives for any ring Λ. Our aim now is to prove:

Theorem (Baer) For any ring Λ, Λ-Mod has enough injectives.

Def^n An object $G \in \mathcal{C}$ is a generator if $\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \longrightarrow \underline{\mathrm{Set}}$ is faithful, that is, whenever $u, v : X \longrightarrow Y$ are distinct (X, Y arbitrary) there is a morphism $g : G \longrightarrow X$ with $u \circ g \neq v \circ g$ which "witnesses" the distinction.

<u>Def^n</u> An object $U \in \mathcal{C}$ is a <u>cogenerator</u> if it is a generator in $\mathcal{C}^{op}$, i.e. whenever $u, v : X \longrightarrow Y$ are distinct there exists $g : Y \longrightarrow G$ with $g \circ u \neq g \circ v$.

<u>Remarks</u> (1) If $\mathcal{C} = \Lambda\text{-Mod}$ then G is a generator $\iff \mathrm{Hom}_{\mathcal{C}}(G, -)$ sends nonzero objects to nonzero objects, and dually.

(2) Λ is a generator for Λ-Mod.

<u>Lemma</u> $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for <u>Ab</u>.

<u>Proof</u> Let A be a nonzero abelian group, and $0 \neq a \in A$. Consider the map

$$\varphi_a : \mathbb{Z} \longrightarrow A, \quad \varphi_a(1) = a.$$

If a has infinite order φ_a induces $\mathbb{Z} \cong (a)$. Define $\varphi : (a) \longrightarrow \mathbb{Q}/\mathbb{Z}$ by choosing any nonzero $\varphi(a)$. If a has order n, then φ_a induces $\mathbb{Z}/n\mathbb{Z} \cong (a)$ and $\varphi : (a) \longrightarrow \mathbb{Q}/\mathbb{Z}$, $\varphi(a) = \frac{1}{n}$ is well-defined. In either case, since $\mathbb{Q}/\mathbb{Z}$ is injective φ lifts to $\psi : A \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $\psi(a) \neq 0$, so $\psi \neq 0$, and we have shown $\mathrm{Hom}_{\underline{Ab}}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$. $\square$

<u>Lemma</u> If G is a projective generator for $\mathcal{C}$ then for any object X there is an epimorphism $\coprod_{i \in I} G \longrightarrow X$ for some set I.

<u>Proof</u> Take $I = \mathrm{Hom}_{\mathcal{C}}(G, X)$ and for $i \in I$ set $G_i := G$ and $f_i : G \to X$ to be i, that is, $f_i := i$. This induces $f : \coprod_{i \in I} G \longrightarrow X$ with $f \circ u_i = f_i$ for all i (here $u_i : G_i \longrightarrow \coprod_{i \in I} G$ are the morphisms into the coproduct). To see f is epi, suppose $af = bf$. Then precomposing with u_i we have $af_i = bf_i$ for all i. But then $a = b$ since G is a generator. $\square$

<u>Lemma</u> If $\mathcal{C}$ has a projective generator it has enough projectives.

Dualising, we have

<u>Lemma</u> If $\mathcal{C}$ has a cogenerator U then for every object X there is a monomorphism $X \longrightarrow \prod_{i \in I} U$ for some index set I.

<u>Lemma</u> If $\mathcal{C}$ has an injective cogenerator it has enough injectives.

<u>Corollary</u> <u>Ab</u> has enough injectives.

To prove Baer's theorem recall:

actually in the end I didn't use this, and stated things directly, but this lemma is still worth keeping

<u>Lemma</u> A Λ-module I is injective if and only if for every exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A'', I) \longrightarrow \mathrm{Hom}_\Lambda(A, I) \longrightarrow \mathrm{Hom}_\Lambda(A', I) \longrightarrow 0 \qquad (*)$$

is exact.

<u>Proof</u> The exactness of $(*)$ is equivalent to $\mathrm{Hom}_\Lambda(A, I) \longrightarrow \mathrm{Hom}_\Lambda(A', I)$ being epi. If I is injective this is clearly true, and if $\circledast$ is exact for every sequence and we are given a mono $A' \xrightarrow{\alpha} A$, apply this to $0 \to A' \to A \to \mathrm{Coker}(\alpha) \to 0$. $\square$

<u>Remark</u> Let G be an abelian group, Λ a ring. Then $\mathrm{Hom}_{\underline{Ab}}(\Lambda, G)$ is naturally a left Λ-module via the action, for $\lambda \in \Lambda$ and $\varphi \in \mathrm{Hom}_{\underline{Ab}}(\Lambda, G)$, $(\lambda \cdot \varphi)(\mu) = \varphi(\mu\lambda)$. This defines a functor $\mathrm{Hom}_{\underline{Ab}}(\Lambda, -) : \underline{Ab} \to \Lambda\text{-Mod}$.

<u>Lemma</u> The functor $\mathrm{Hom}_{\underline{Ab}}(\Lambda, -)$ is right adjoint to the forgetful functor $F : \Lambda\text{-Mod} \longrightarrow \underline{Ab}$.

<u>Proof</u> Define a function, for a Λ-module M and abelian group G,

$$\Phi : \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_{\underline{Ab}}(\Lambda, G)) \longrightarrow \mathrm{Hom}_{\underline{Ab}}(F(M), G) \qquad (4.1)$$

by the formula $\Phi(f)(m) = f(m)(1_\Lambda)$. Since f is additive, $\Phi(f)$ is an element of $\mathrm{Hom}_{\underline{Ab}}(FM, G)$. We define

$$\Phi' : \mathrm{Hom}_{\underline{Ab}}(FM, G) \longrightarrow \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_{\underline{Ab}}(\Lambda, G))$$

by $\Phi'(g)(m)(\lambda) = g(\lambda m)$. Again it is clear $\Phi'(g)(m)$ is linear in λ, and that $\Phi'(g)$ is additive in m. To see $\Phi'(g)$ is Λ-linear, observe for $\mu \in \Lambda$

$$\begin{aligned} \Phi'(g)(\mu m)(\lambda) &= g(\lambda \mu m) \\ &= (\mu \cdot \Phi'(g)(m))(\lambda) \end{aligned}$$

so $\Phi'(g)(\mu m) = \mu \cdot \Phi'(g)(m)$ as required. We need only show Φ is natural (which we leave as an easy exercise) and that $\Phi' \circ \Phi = id$, $\Phi \circ \Phi' = id$.

$\boxed{\Phi' \circ \Phi = id}$ $$\begin{aligned} \Phi'(\Phi(f))(m)(\lambda) &= \Phi(f)(\lambda m) = f(\lambda m)(1_\Lambda) \\ &= (\lambda \cdot f(m))(1_\Lambda) = f(m)(\lambda) \quad \therefore \Phi'(\Phi f) = f \end{aligned}$$

$\boxed{\Phi \circ \Phi' = id}$ $\Phi(\Phi' g)(m) = \Phi' g(m)(1_\Lambda) = g(m)$ so $\Phi(\Phi' g) = g$. $\square$

<u>Ex 3</u> Check Φ is natural in M, G.

(The reference for today's lecture is Mitchell "Theory of categories" II.15)

Notice that (4.1) is an isomorphism of abelian groups, not just sets.

<u>Proof of Baer's theorem</u> Take $G = \mathbb{Q}/\mathbb{Z}$ in the above to get a natural iso

$$\mathrm{Hom}_\Lambda(M, \mathrm{Hom}_{\underline{Ab}}(\Lambda, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\underline{Ab}}(FM, \mathbb{Q}/\mathbb{Z}). \qquad (*)$$

Set $E := \mathrm{Hom}_{\underline{Ab}}(\Lambda, \mathbb{Q}/\mathbb{Z})$. We claim this is an injective cogenerator for Λ-Mod, whence Λ-Mod has enough injectives, and we are done.

- E is a cogenerator since for a Λ-module M, by $(*)$

$$M = 0 \underset{\uparrow \ \mathbb{Q}/\mathbb{Z} \text{ cogenerates } \underline{Ab}}{\iff} \mathrm{Hom}_{\underline{Ab}}(FM, \mathbb{Q}/\mathbb{Z}) = 0 \iff \mathrm{Hom}_\Lambda(M, E) = 0.$$

- E is injective. Since $(*)$ is natural, for a mono $u : M \longrightarrow M'$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(M', E) \arrow[r, "u^*"] \arrow[d, "\cong"] & \mathrm{Hom}_\Lambda(M, E) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\underline{Ab}}(M', \mathbb{Q}/\mathbb{Z}) \arrow[r, "u^*"'] & \mathrm{Hom}_{\underline{Ab}}(M, \mathbb{Q}/\mathbb{Z}).
\end{tikzcd}$$

Since $\mathbb{Q}/\mathbb{Z}$ is injective the bottom row is surjective, hence so is the top row, whence E is injective. $\square$